

# ON THETA FUNCTIONS

BERNDT E. SCHWERDTFEGER

ABSTRACT. Definition and properties of JACOBI's  $\vartheta$ -function.

## PREFACE

In this paper I present the foundation of JACOBI's  $\vartheta$ -functions, based on his *Notices sur les fonctions elliptiques* [3, vol. I, 7.] and his lecture *Theorie der elliptischen Functionen* [3, vol. I, 19.]. I derive all his  $\vartheta$ -relations, in particular his *merkwürdige Relation of theta-constants*

$$\vartheta_{00}^4(0, \tau) = \vartheta_{01}^4(0, \tau) + \vartheta_{10}^4(0, \tau)$$

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## 1. JACOBI'S THETA SERIES

The source of JACOBI's  $\vartheta$ -functions is in his *Notices sur les fonctions elliptiques* [3, vol. I, 7.] (CRELLE, 1828), as well as his *Fundamenta nova theoriae functionum ellipticarum* ([3, vol. I, 4.], 1829).

Later JACOBI reversed the development and started with the theta series to derive the theory of *elliptic* functions in his lecture *Theorie der elliptischen Functionen, aus den Eigenschaften der Thetareihen abgeleitet* prepared by BORCHARDT in 1838 on behalf of JACOBI [3, vol. I, 19.]. We will take a rapid walk through the first part of JACOBI's lecture. For his notations and comparison with later authors see the section 5.

Let  $\mathbb{D} = \{x \in \mathbb{C} \mid |x| < 1\}$  be the open unit disk in  $\mathbb{C}$ . The complex line  $\mathbb{C}$  is the universal covering of  $\mathbb{C}^\times = \mathbb{C} - \{0\}$  via the exponential map

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \xrightarrow{e} \mathbb{C}^\times \rightarrow 1$$

where  $t = \mathbf{e}(x) = \exp(2\pi i x) \in \mathbb{C}^\times$  for  $x \in \mathbb{C}$ . In the following diagram

$$\begin{array}{ccc} \mathbb{H} & \hookrightarrow & \mathbb{C} \\ \mathbf{e} \downarrow & & \downarrow \mathbf{e} \\ \mathbb{D}^\times & \hookrightarrow & \mathbb{C}^\times \end{array}$$

$\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\} = \mathbf{e}^{-1}(\mathbb{D}^\times)$ , the upper half plane, is the universal covering of the punctured disk  $\mathbb{D}^\times = \mathbb{D} - \{0\}$ . Variables are denoted  $x \in \mathbb{C}$  and  $t = \mathbf{e}(x) \in \mathbb{C}^\times$ , resp.  $\tau \in \mathbb{H}$  and (*sic!*)  $q = \mathbf{e}(\tau/2) \in \mathbb{D}^\times$ .

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## 2. DEFINITIONS AND PROPERTIES

Throughout this note I will use the following definition for  $\vartheta$

**Definition 2.1.** Let  $\vartheta : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$  be given by:

$$\vartheta(x, \tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}(n^2 \tau / 2 + nx) = \sum_{n \in \mathbb{Z}} q^{n^2} \cdot t^n$$

*Remark.* In JACOBI's notation  $\vartheta_3(\pi x, q) = \vartheta(x, \tau)$ ,  $q = \mathbf{e}(\tau/2)$ , see section 5.

**Proposition 2.1.** *The series for  $\vartheta$  converges absolutely and uniformly on compact subsets, defining an analytic function*

$$\begin{aligned} \vartheta : \mathbb{C} \times \mathbb{H} &\longrightarrow \mathbb{C} \\ (x, \tau) &\longmapsto \vartheta(x, \tau) \end{aligned}$$

*Proof.* Let  $K \subset \mathbb{C} \times \mathbb{H}$  be compact, then there is  $m, M \in \mathbb{R}$ ,  $m, M > 0$  such that for all  $(x, \tau) \in K$ ,  $\text{Im } \tau \geq m$  and  $-M \leq \text{Im } x \leq M$ . Now let  $q_0 = e^{-\pi m}$  and  $t_0 = e^{2\pi M}$ , then  $|q| = |\mathbf{e}(\tau/2)| = \exp(-\pi \text{Im } \tau) \leq q_0$  and  $t_0^{-1} \leq |t| \leq t_0$ . We then have

$$|\mathbf{e}(\frac{\tau}{2}n^2 + nx)| = |q|^{n^2} \cdot |t|^n \leq \begin{cases} q_0^{n^2} \cdot t_0^n & n \geq 0 \\ q_0^{n^2} \cdot t_0^{-n} & n < 0 \end{cases}$$

Take an integer  $n_0 > \frac{2M}{m}$ , then  $q_1 = q_0^{n_0} t_0 < 1$  and for  $n \geq n_0$  we have  $q_0^n t_0 \leq q_1$ , hence  $q_0^{n^2} \cdot t_0^n = (q_0^n t_0)^n \leq q_1^n$ . Similarly for  $n \leq -n_0$ , since  $-n \geq n_0$ ,  $q_0^{n^2} \cdot t_0^{-n} \leq q_1^{-n}$ . Hence, the series is majorized by a geometric series on the compact set  $K$ .  $\square$

This proof of convergence can easily be adopted to the situation where we sum over a shifted set  $a + \mathbb{Z}$ ,  $a \in \mathbb{C}$ , instead of  $\mathbb{Z}$  in the sum defining  $\vartheta$ , leading to the

**Definition 2.2.** For  $a, b \in \mathbb{C}$  the *shifted*  $\vartheta$  is defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (x, \tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}(\frac{\tau}{2}(n+a)^2 + (n+a)(x+b)) = \sum_{n \in a+\mathbb{Z}} \mathbf{e}(\frac{\tau}{2}n^2 + n(x+b))$$

So, in particular,  $\vartheta(x, \tau) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (x, \tau)$  and  $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (x, \tau) = \vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (x+b, \tau)$ .

Direct calculation yields the equation

$$(1) \quad \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (x, \tau) = \mathbf{e}(\frac{\tau}{2}a^2 + a(x+b)) \cdot \vartheta(x+a\tau+b, \tau),$$

which implies

$$(2) \quad \vartheta \begin{bmatrix} a+m \\ b+n \end{bmatrix} (x, \tau) = \mathbf{e}(an) \cdot \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (x, \tau) \quad \text{for } m, n \in \mathbb{Z},$$

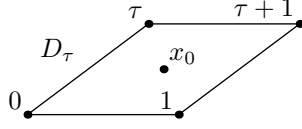
and in particular

$$(3) \quad \vartheta(x+1, \tau) = \vartheta(x, \tau),$$

$$(4) \quad \vartheta(x+\tau, \tau) = \mathbf{e}(-\frac{\tau}{2} - x) \cdot \vartheta(x, \tau) = q^{-1} t^{-1} \vartheta(x, \tau).$$

We note that  $\vartheta$  has a zero at  $x_0 = (1+\tau)/2$ , as

$$\vartheta(x_0, \tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}(\frac{\tau}{2}n^2 + n\frac{1+\tau}{2}) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(n+1)} = 0.$$



It is in fact the only zero in a fundamental domain  $D_\tau$  of the lattice  $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z} \subset \mathbb{C}$ . This follows from the residue theorem: the number of zeros with multiplicity is

$$\sum_{x \in D_\tau} \text{ord}_x \vartheta(x, \tau) = \sum_{x \in D_\tau} \text{Res}\left(\frac{\vartheta'}{\vartheta}, x\right) = \frac{1}{2\pi i} \int_\gamma \frac{\vartheta'(x, \tau)}{\vartheta(x, \tau)} dx$$

where  $\gamma = \partial D_\tau$  is the boundary of  $D_\tau$ . We evaluate the integral over the path  $\gamma$ :

$$\int_\gamma \frac{\vartheta'(x, \tau)}{\vartheta(x, \tau)} dx = \int_0^1 \left( \frac{\vartheta'(x, \tau)}{\vartheta(x, \tau)} - \frac{\vartheta'(x + \tau, \tau)}{\vartheta(x + \tau, \tau)} \right) dx + \int_0^\tau \left( \frac{\vartheta'(x + 1, \tau)}{\vartheta(x + 1, \tau)} - \frac{\vartheta'(x, \tau)}{\vartheta(x, \tau)} \right) dx$$

and the last integral is 0 because of the periodicity (3). The logarithmic derivative of (4) gives the relation

$$\frac{\vartheta'(x + \tau, \tau)}{\vartheta(x + \tau, \tau)} = -2\pi i + \frac{\vartheta'(x, \tau)}{\vartheta(x, \tau)}$$

and hence  $\sum \text{ord}_x \vartheta(x, \tau) = 1$ , so there is exactly one *simple* zero of  $\vartheta(x, \tau)$  at  $x_0 \in D_\tau$ .  $\square$

For  $a, b \in \frac{1}{2}\mathbb{Z}$  the following special notation is used.

**Definition 2.3.** For  $a, b \in \{0, 1\}$  define  $\vartheta_{ab} = \vartheta \begin{bmatrix} a/2 \\ b/2 \end{bmatrix}$ .

By (1) this amounts to

$$\vartheta_{00}(x, \tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}\left(\frac{\tau}{2}n^2 + nx\right) = \vartheta(x, \tau)$$

$$\vartheta_{01}(x, \tau) = \sum_{n \in \mathbb{Z}} (-1)^n \mathbf{e}\left(\frac{\tau}{2}n^2 + nx\right) = \vartheta\left(x + \frac{1}{2}, \tau\right)$$

$$\vartheta_{10}(x, \tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}\left(\frac{\tau}{2}\left(n + \frac{1}{2}\right)^2 + \left(n + \frac{1}{2}\right)x\right) = \mathbf{e}\left(\frac{\tau}{8}\right) \mathbf{e}\left(\frac{x}{2}\right) \vartheta\left(x + \frac{\tau}{2}, \tau\right)$$

$$\vartheta_{11}(x, \tau) = \sum_{n \in \mathbb{Z}} i(-1)^n \mathbf{e}\left(\frac{\tau}{2}\left(n + \frac{1}{2}\right)^2 + \left(n + \frac{1}{2}\right)x\right) = i \mathbf{e}\left(\frac{\tau}{8}\right) \mathbf{e}\left(\frac{x}{2}\right) \vartheta\left(x + \frac{1 + \tau}{2}, \tau\right)$$

We list a table of half period values<sup>1</sup>  $\vartheta_{ab}(x + \lambda)$  for a  $\lambda \in \frac{1}{2}\Lambda_\tau$  and where  $\varepsilon = \varepsilon(x, \tau) = \mathbf{e}\left(-\frac{x}{2} - \frac{\tau}{8}\right)$  is an exponential factor:

$$\begin{array}{lll} \vartheta_{00}\left(x + \frac{1}{2}\right) = \vartheta_{01}(x) & \vartheta_{00}\left(x + \frac{\tau}{2}\right) = \varepsilon \vartheta_{10}(x) & \vartheta_{00}\left(x + \frac{1 + \tau}{2}\right) = -i\varepsilon \vartheta_{11}(x) \\ \vartheta_{01}\left(x + \frac{1}{2}\right) = \vartheta_{00}(x) & \vartheta_{01}\left(x + \frac{\tau}{2}\right) = -i\varepsilon \vartheta_{11}(x) & \vartheta_{01}\left(x + \frac{1 + \tau}{2}\right) = \varepsilon \vartheta_{10}(x) \\ \vartheta_{10}\left(x + \frac{1}{2}\right) = \vartheta_{11}(x) & \vartheta_{10}\left(x + \frac{\tau}{2}\right) = \varepsilon \vartheta_{00}(x) & \vartheta_{10}\left(x + \frac{1 + \tau}{2}\right) = -i\varepsilon \vartheta_{01}(x) \\ \vartheta_{11}\left(x + \frac{1}{2}\right) = -\vartheta_{10}(x) & \vartheta_{11}\left(x + \frac{\tau}{2}\right) = -i\varepsilon \vartheta_{01}(x) & \vartheta_{11}\left(x + \frac{1 + \tau}{2}\right) = -\varepsilon \vartheta_{00}(x) \end{array}$$

these follow from the definitions and (3) and (4) (cf. [3, vol. I, 19., (2.), p. 502]). For completeness we also list the equations corresponding to (3) and (4):

$$\begin{array}{ll} \vartheta_{01}(x + 1) = \vartheta_{01}(x) & \vartheta_{01}(x + \tau) = -q^{-1}t^{-1}\vartheta_{01}(x) \\ \vartheta_{10}(x + 1) = -\vartheta_{10}(x) & \vartheta_{10}(x + \tau) = q^{-1}t^{-1}\vartheta_{10}(x) \\ \vartheta_{11}(x + 1) = -\vartheta_{11}(x) & \vartheta_{11}(x + \tau) = -q^{-1}t^{-1}\vartheta_{11}(x) \end{array}$$

<sup>1</sup>dropping  $\tau$  from the notation

## 3. DERIVING A REMARKABLE RELATION

In his 1838 lecture JACOBI proceeds to derive several formulas between sums of products of four  $\vartheta$ -series. To lighten the notation, let us agree that  $\vartheta(x) = \vartheta(x_1)\vartheta(x_2)\vartheta(x_3)\vartheta(x_4)$  for vectors  $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$  and similarly for the other thetas.

JACOBI considers the linear reflection at the hyperplane  $x_1 - x_2 - x_3 - x_4 = 0$

$$\begin{aligned} \mathbb{C}^4 &\longrightarrow \mathbb{C}^4 \\ x &\longmapsto x' = x \cdot A \end{aligned}$$

given by the matrix

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

which satisfies  $A = {}^t A$  and  $A^2 = 1$ . His *Fundamentalsatz* [3, vol. I, 19.2.,(11.)] is the relation (5) in the next theorem:

**Theorem 3.1.**

$$(5) \quad \vartheta_{00}(x) + \vartheta_{10}(x) = \vartheta_{00}(x') + \vartheta_{10}(x')$$

$$(6) \quad \vartheta_{00}(x) - \vartheta_{10}(x) = \vartheta_{01}(x') + \vartheta_{11}(x')$$

$$(7) \quad \vartheta_{01}(x) + \vartheta_{11}(x) = \vartheta_{00}(x') - \vartheta_{10}(x')$$

$$(8) \quad \vartheta_{01}(x) - \vartheta_{11}(x) = \vartheta_{01}(x') - \vartheta_{11}(x')$$

*Proof.* JACOBI's reasoning rests on the observation that for  $x, y \in \mathbb{C}^4$  the bilinear form  $x \cdot {}^t y = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$  is invariant under the involution  $x \mapsto x'$ :

$$x' \cdot {}^t y' = x \cdot A \cdot {}^t (y \cdot A) = x \cdot A \cdot {}^t A \cdot {}^t y = x \cdot {}^t y$$

With our convention for  $x \in \mathbb{C}^4$  we note that by definition

$$\vartheta_{00}(x) = \sum_{n \in \mathbb{Z}^4} \mathbf{e}(\frac{\tau}{2} n \cdot {}^t n + n \cdot {}^t x) \quad \vartheta_{10}(x) = \sum_{n \in (\frac{1}{2} + \mathbb{Z})^4} \mathbf{e}(\frac{\tau}{2} n \cdot {}^t n + n \cdot {}^t x)$$

JACOBI writes down the formulas relating  $n = (a, b, c, d)$  and  $n' = (a', b', c', d')$

$$\begin{aligned} a' &= \frac{1}{2}(a + b + c + d) & b' &= \frac{1}{2}(a + b - c - d) \\ c' &= \frac{1}{2}(a - b + c - d) & d' &= \frac{1}{2}(a - b - c + d) \end{aligned}$$

and for  $n \in \mathbb{Z}^4 \cup (\frac{1}{2} + \mathbb{Z})^4$  he emphasizes that the numbers

$$a' + b' = a + b \quad a' + c' = a + c \quad a' + d' = a + d$$

are integers, which implies  $n' \in \mathbb{Z}^4 \cup (\frac{1}{2} + \mathbb{Z})^4$ , hence the involution induces a bijection  $\mathbb{Z}^4 \cup (\frac{1}{2} + \mathbb{Z})^4 \xrightarrow{\sim} \mathbb{Z}^4 \cup (\frac{1}{2} + \mathbb{Z})^4$  on the index set. Together with the invariances  $n \cdot {}^t n = n' \cdot {}^t n'$ ,  $n \cdot {}^t x = n' \cdot {}^t x'$ , the relation (5) becomes obvious.

Applying (5) to  $(x_1 + 1, x_2, x_3, x_4)' = (x'_1 + \frac{1}{2}, x'_2 + \frac{1}{2}, x'_3 + \frac{1}{2}, x'_4 + \frac{1}{2})$  and making use of the table of half period values we get (6). (7) is identical with (6), by interchanging  $x$  with  $x'$ , as  $x'' = x$ . (8) is obtained from (6) by applying the reverse operation  $(x_1 + \frac{1}{2}, x_2 + \frac{1}{2}, x_3 + \frac{1}{2}, x_4 + \frac{1}{2})' = (x'_1 + 1, x'_2, x'_3, x'_4)$ .  $\square$

The system of equations (5)–(8) can be combined into one vector equation

$$(9) \quad (\vartheta_{00}(x), \vartheta_{01}(x), \vartheta_{10}(x), \vartheta_{11}(x))' = (\vartheta_{00}(x'), \vartheta_{01}(x'), \vartheta_{10}(x'), \vartheta_{11}(x'))$$

*Remark.* Of course, (9) and the system of equations (5)–(8) are equivalent. The vector components of (9) correspond to the equations (R2)–(R5) of MUMFORD in [5, I, §5], whereas (5)–(8) are identical to JACOBI’s table (A.): (1.)–(4.) in [3, vol. I, 19., p. 507].

In the sequel JACOBI substitutes various different vectors into (9). I skip some of them and only list the outcome for the vector  $(x, x, y, y)' = (x + y, x - y, 0, 0)$

$$\begin{aligned}\vartheta_{00}(x+y)\vartheta_{00}(x-y)\vartheta_{00}^2(0) &= \vartheta_{00}^2(x)\vartheta_{00}^2(y) + \vartheta_{11}^2(x)\vartheta_{11}^2(y) = \\ &= \vartheta_{01}^2(x)\vartheta_{01}^2(y) + \vartheta_{10}^2(x)\vartheta_{10}^2(y) \\ \vartheta_{01}(x+y)\vartheta_{01}(x-y)\vartheta_{01}^2(0) &= \vartheta_{00}^2(x)\vartheta_{00}^2(y) - \vartheta_{10}^2(x)\vartheta_{10}^2(y) = \\ &= \vartheta_{01}^2(x)\vartheta_{01}^2(y) - \vartheta_{11}^2(x)\vartheta_{11}^2(y) \\ \vartheta_{10}(x+y)\vartheta_{10}(x-y)\vartheta_{10}^2(0) &= \vartheta_{00}^2(x)\vartheta_{00}^2(y) - \vartheta_{01}^2(x)\vartheta_{01}^2(y) = \\ &= \vartheta_{10}^2(x)\vartheta_{10}^2(y) - \vartheta_{11}^2(x)\vartheta_{11}^2(y)\end{aligned}$$

For  $y = x$  in particular

$$\vartheta_{00}(2x)\vartheta_{00}^3(0) = \vartheta_{00}^4(x) + \vartheta_{11}^4(x) = \vartheta_{01}^4(x) + \vartheta_{10}^4(x)$$

whereas for  $y = 0$  it gives  $\vartheta_{00}^2(x)\vartheta_{00}^2(0) = \vartheta_{01}^2(x)\vartheta_{01}^2(0) + \vartheta_{10}^2(x)\vartheta_{10}^2(0)$ . Substituting  $x \mapsto x + \frac{1}{2} + \frac{\tau}{2}$  yields  $-\varepsilon^2\vartheta_{11}^2(x)\vartheta_{00}^2(0) = \varepsilon^2\vartheta_{10}^2(x)\vartheta_{01}^2(0) - \varepsilon^2\vartheta_{01}^2(x)\vartheta_{10}^2(0)$ , hence

$$\begin{aligned}\vartheta_{00}^2(x)\vartheta_{00}^2(0) &= \vartheta_{01}^2(x)\vartheta_{01}^2(0) + \vartheta_{10}^2(x)\vartheta_{10}^2(0) \\ \vartheta_{11}^2(x)\vartheta_{00}^2(0) &= \vartheta_{01}^2(x)\vartheta_{10}^2(0) - \vartheta_{10}^2(x)\vartheta_{01}^2(0)\end{aligned}$$

Finally for  $x = 0$  we obtain the *remarkable relation* (in JACOBI [3, vol. I, 19., (E.) p. 511] *die merkwürdige Relation*) for the *Theta-Nullwerte*

$$(10) \quad \vartheta_{00}^4(0, \tau) = \vartheta_{01}^4(0, \tau) + \vartheta_{10}^4(0, \tau)$$

i.e.

$$(1+2q+2q^4+2q^9+\dots)^4 = (1-2q+2q^4-2q^9+\dots)^4 + 16q(1+q^{1-2}+q^{2-3}+q^{3-4}+\dots)^4$$

#### 4. VARIATION WITH THE *module* $\tau$

In the previous section we have kept the *module*  $\tau$  fixed (and sometimes dropped it from the notation). We are now proving the behaviour of  $\vartheta$  with respect to variation of  $\tau$ .

We start by looking at  $\vartheta(x, \tau + 1)$ .

$$\begin{aligned}\vartheta(x, \tau + 1) &= \sum \mathbf{e}\left(\frac{\tau+1}{2}n^2 + nx\right) = \sum \mathbf{e}\left(\frac{\tau}{2}n^2 + \frac{1}{2}n^2 + nx\right) = \\ &= \sum (-1)^n \mathbf{e}\left(\frac{\tau}{2}n^2 + nx\right) = \vartheta_{01}(x, \tau) = \vartheta(x + 1/2, \tau)\end{aligned}$$

where we used  $(-1)^{n^2} = (-1)^n$ . In particular,  $\vartheta(x, \tau + 2) = \vartheta(x, \tau)$ .

Next we are going to look at  $\vartheta(x, -1/\tau)$ . JACOBI states in [3, vol. I, 7., p. 264] the formula

$$\mathbf{H}(ix, q) = i\sqrt{\frac{K}{K'}} \exp\left(\frac{Kxx}{\pi K'}\right) \mathbf{H}\left(\frac{Kx}{K'}, q'\right)$$

where  $q = \exp(\frac{-\pi K'}{K})$  and  $q' = \exp(\frac{-\pi K}{K'})$ . With  $\tau = iK'/K$  (such that  $q = \mathbf{e}(\tau/2)$ ) this can be rewritten (see section 5) in our notation as

$$\vartheta_{11}(x, \tau) = i\sqrt{i/\tau} \exp(-\pi ix^2/\tau) \vartheta_{11}(x/\tau, -1/\tau)$$

This can be transformed into the following set of equivalent equations:

**Theorem 4.1.**

$$\begin{aligned}
\vartheta(x, -1/\tau) &= \sqrt{\tau/i} \mathbf{e}(x^2\tau/2)\vartheta(x\tau, \tau) \\
\vartheta_{01}(x, -1/\tau) &= \sqrt{\tau/i} \mathbf{e}(x^2\tau/2)\vartheta_{10}(x\tau, \tau) \\
\vartheta_{10}(x, -1/\tau) &= \sqrt{\tau/i} \mathbf{e}(x^2\tau/2)\vartheta_{01}(x\tau, \tau) \\
\vartheta_{11}(x, -1/\tau) &= -i\sqrt{\tau/i} \mathbf{e}(x^2\tau/2)\vartheta_{11}(x\tau, \tau)
\end{aligned}$$

*Proof.* The equivalence follows from the table of half period values. We are going to prove the first one.

Recall some FOURIER transforms:  $\exp(-\pi x^2)$  is its own transform, hence, for  $t > 0$ ,  $g(x) = \exp(-\pi t x^2)$  has the transform  $\widehat{g}(x) = \frac{1}{\sqrt{t}} \exp(-\frac{\pi}{t} x^2)$  and for  $h(x) = g(x+a)$  we get  $\widehat{h}(x) = \int g(t+a)\mathbf{e}(tx)dt = \int g(t)\mathbf{e}((t-a)x)dt = \widehat{g}(x)\mathbf{e}(-ax)$ .

The POISSON formula  $\sum h(n) = \sum \widehat{h}(n)$  now yields the equation

$$(11) \quad \sum_n \exp(-\pi t(n+a)^2) = \frac{1}{\sqrt{t}} \sum_n \exp(-\frac{\pi}{t} n^2) \mathbf{e}(-an)$$

Now by (1)  $\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (0, \tau) = \mathbf{e}(\frac{\tau}{2}a^2)\vartheta(a\tau, \tau) = \sum \mathbf{e}(\frac{\tau}{2}(n+a)^2)$ , which is the left hand side of our POISSON equation (11) for  $\tau = ti$ , whereas the sum on the right hand side is  $\sum \mathbf{e}(-\frac{n^2}{2\tau} - an) = \vartheta(-a, -1/\tau) = \vartheta(a, -1/\tau)$  and equation (11) reads

$$\mathbf{e}(\frac{\tau}{2}a^2)\vartheta(a\tau, \tau) = \sqrt{i/\tau}\vartheta(a, -1/\tau)$$

which, by analytic continuation, holds for all  $\tau \in \mathbb{H}$ .  $\square$

## 5. NOTATION FOR THETA FUNCTIONS BY DIFFERENT AUTHORS

The notation in the literature varies. JACOBI himself used different notations for his  $\vartheta$ -functions at various times. In his *Notices sur les fonctions elliptiques* in 1828 in Crelle's Journal [3, vol. I, 7.], JACOBI introduced the notation H (Eta) and  $\Theta$  (Theta) (loc.cit. p. 256) for the numerator resp. denominator of his *sinus amplitudinis*

$$\begin{aligned}
\sin \operatorname{am} \frac{2Kx}{\pi} &= \frac{1}{\sqrt{k}} \frac{H(x)}{\Theta(x)} \\
H(x) &= 2\sqrt[4]{q} \sin x - 2\sqrt[4]{q^9} \sin 3x + 2\sqrt[4]{q^{25}} \sin 5x - 2\sqrt[4]{q^{49}} \sin 7x + \dots \\
\Theta(x) &= 1 - 2q \cos 2x + 2q^4 \cos 4x - 2q^9 \cos 6x + 2q^{16} \cos 8x - \dots
\end{aligned}$$

A year later in his book *Fundamenta Nova Theoriae Functionum Ellipticarum* [3, vol. I, 4., p. 198, p. 224] he used  $H(\frac{2Kx}{\pi}) = H(x)$  and  $\Theta(\frac{2Kx}{\pi}) = \Theta(x)$  instead (see the remarks of WEIERSTRASS on p. 542). In the 1838 lecture they are called  $\vartheta_1(x) = H(x)$  resp.  $\vartheta(x) = \Theta(x)$ , where his definitions are as follows:

For  $q \in \mathbb{C}^\times$  such that  $|q| < 1$  JACOBI defines (loc.cit. p. 501):

$$\begin{aligned}
\vartheta(x, q) &= \sum (-1)^\nu q^{\nu^2} e^{2\nu xi} & \vartheta_1(x, q) &= -\sum i^{2\nu+1} q^{\frac{1}{4}(2\nu+1)^2} e^{(2\nu+1)xi} \\
\vartheta_2(x, q) &= \sum q^{\frac{1}{4}(2\nu+1)^2} e^{(2\nu+1)xi} & \vartheta_3(x, q) &= \sum q^{\nu^2} e^{2\nu xi}
\end{aligned}$$

Later authors introduced other fashions, like  $\theta$  versus  $\vartheta$ , and ever changing subscripts. Here is a translation table of notations by selected mathematicians.

JACOBI Notices, 1828	$\Theta(\pi x)$	$H(\pi x)$		
JACOBI Fundamenta nova, 1829	$\Theta(2Kx)$	$H(2Kx)$		
JACOBI Lecture notes, 1838	$\vartheta(\pi x, q)$	$\vartheta_1(\pi x, q)$	$\vartheta_2(\pi x, q)$	$\vartheta_3(\pi x, q)$
WEIERSTRASS	$\vartheta_0(x \tau)$	$\vartheta_1(x \tau)$	$\vartheta_2(x \tau)$	$\vartheta_3(x \tau)$
HERMITE	$\theta_{0,1}(x, \tau)$	$-i\theta_{1,1}(x, \tau)$	$\theta_{1,0}(x, \tau)$	$\theta_{0,0}(x, \tau)$
C. JORDAN	$\theta_2(x, \tau)$	$\theta(x, \tau)$	$\theta_1(x, \tau)$	$\theta_3(x, \tau)$
H. CARTAN	$\vartheta_0(x, \tau)$	$\vartheta_1(x, \tau)$		
MUMFORD	$\vartheta_{01}(x, \tau)$	$-\vartheta_{11}(x, \tau)$	$\vartheta_{10}(x, \tau)$	$\vartheta_{00}(x, \tau)$

WEIERSTRASS in *Einführung der Thetafunctionen* [6, §34.] describes the relation to JACOBI and HERMITE, who defined  $\theta_{\mu,\nu}(x) = \sum_m (-1)^{m\nu} e^{(\frac{\tau}{2}(m + \frac{\mu}{2})^2 + (m + \frac{\mu}{2})x)}$ . CARTAN follows WEIERSTRASS in [1, chap. V, ex. 3]. JORDAN introduced *Les fonctions*  $\theta(x, \tau)$ ,  $\theta_1(x, \tau)$ ,  $\theta_2(x, \tau)$ ,  $\theta_3(x, \tau)$  (V, n<sup>o</sup> 426) in *Fonctions elliptiques* [4, chap. VII] and relates their difference to WEIERSTRASS' notation in n<sup>o</sup> 427. CHANDRASEKHARAN uses this notation in [2, V, §8] of JORDAN's, as does WEIL in [7, chap. IV, §8] for  $\theta(\zeta, \tau)$ .

I have chosen MUMFORD's notation in [5].

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