SPHERICAL DESIGN WITH METAPOST

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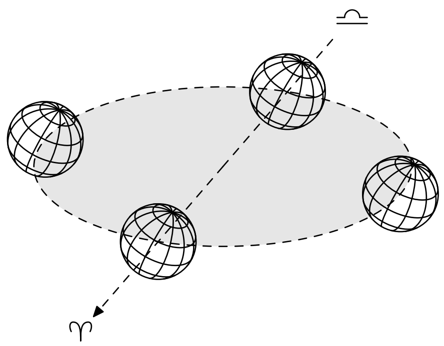
For Thomas

竹 There are whole bamboos in his heart.



PREFACE

When reading ROEGEL [4] I wondered how the drawings were designed. Working at my implementation I realized that quite effective METAPOST-code is achieved by a conceptual mathematical approach. Application to Astronomy completes this paper.

Berlin, November 10, 2011

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1. SPHERE DRAWINGS

DENIS ROEGEL [4] explains how to draw correctly *orthogonal projections* of circles on spheres to some *tangential* plane. By systematic use of Euclidean geometry auxiliary constructions like the orientation of ellipses in [4, 4.7] are avoided and projections of arbitrarily positioned curves of second order are achieved.

1.1. **Orthogonal projection.** Let a 3-dimensional real vector space V be given with Euclidean scalar product $v \cdot w \in \mathbf{R}$ and vector cross product $v \times w \in V$ for two vectors $v, w \in V$. We make frequent use of the formulas:

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w \qquad \text{GRASSMANN} \tag{1}$$

$$(v_1 \times v_2) \cdot (w_1 \times w_2) = \begin{vmatrix} v_1 \cdot w_1 & v_1 \cdot w_2 \\ v_2 \cdot w_1 & v_2 \cdot w_2 \end{vmatrix} \qquad \text{LAGRANGE} \tag{2}$$

The relation $v \cdot w = 0$ says that v and w are *orthogonal*, short $v \perp w$.

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Similarly, $v \perp W$ for a subspace $W \subset V$ signifies $v \perp w$ for all $w \in W$. The *orthogonal complement* to w (resp. W) is the subspace consisting of the vectors $v \perp w$ (resp. $v \perp W$) and is denoted by $w^\perp$ (resp. $W^\perp$). We note $w^\perp = (\mathbf{R}w)^\perp$ and $(rw)^\perp = w^\perp$ for a real $r \neq 0$, therefore we often stick to *unit vectors* w only ($w \cdot w = 1$). The $w^\perp$ are planes for $w \neq 0$ and any plane is of this kind. We call (W, w) an *orthogonal pair* if $w \cdot w = 1$ and $W = w^\perp$. We will often denote the plane with $T_w = w^\perp$ (*tangential plane* at unit vector).

If u, w are *linearly independent*, i.e. $u \times w \neq 0$, we obviously have

$$T_u \cap T_w = \mathbf{R}(u \times w) \qquad\qquad T_{u\times w} = \mathbf{R}u \oplus \mathbf{R}w \tag{3}$$

Proof. From $u, w \perp u \times w$ follows $\mathbf{R}u + \mathbf{R}w \subset T_{u\times w}$ hence equality of the planes. Similarly we get $\mathbf{R}(u \times w) \subset T_u \cap T_w$ and the latter is a line since $T_u \neq T_w$. □

In the *orthogonal decomposition* $V = W \oplus W^\perp$ the projection $p_W : V \to W^\perp$ is called *orthogonal projection* along W. Similarly is $p_w : V \to w^\perp$ the orthogonal projection along w. In a direct sum $V = U \oplus W$, in which $U \perp W$, we have $U^\perp = W$, $W^\perp = U$. For each $v \in V$ holds $v = p_W(v) + p_U(v)$. In a non trivial decomposition ($U, W \neq 0$) one of the subspaces is a line, the other a plane, say $U = \mathbf{R}u$, $u \cdot u = 1$, then we have $p_W(v) = (u \cdot v)u$, $p_u(v) = v - (u \cdot v)u$. Remark, that (1) implies $u \times (v \times u) = v - (u \cdot v)u = p_u(v)$.

Let two different planes T_u, T_w be given. As $T_u \neq T_w$ in particular u, w are *linearly independent*. We have a *canonical decomposition* of the plane T_u (resp. T_w) into lines

$$\begin{aligned} T_u &= (T_u \cap T_w) \oplus (T_u \cap T_{u\times w}) \\ T_w &= (T_u \cap T_w) \oplus (T_w \cap T_{u\times w}) \end{aligned} \tag{4}$$

Proof. From (3) follows $T_u \cap T_w = \mathbf{R}(u \times w)$, $T_u \cap T_{u\times w} = \mathbf{R}(u \times (u \times w))$, and as $u \times w \perp u \times (u \times w)$ the decomposition is even *orthogonal*. □

We have $p_w(T_{u\times w}) \subset T_{u\times w}$, as $v \perp u \times w$ implies by $w \perp u \times w$ also $p_w(v) = v - (u \cdot w)w \perp u \times w$. The orthogonal projection p_w respects (4): it induces the identity on $T_u \cap T_w$ and maps the line $T_u \cap T_{u\times w}$ in T_u into the line $T_w \cap T_{u\times w}$ in T_w, thus is acting by multiplication of a factor. Because

$$p_w(u \times (u \times w)) = p_w((u \cdot w)u - w) = (u \cdot w)p_w(u) = (u \cdot w)(w \times (u \times w))$$

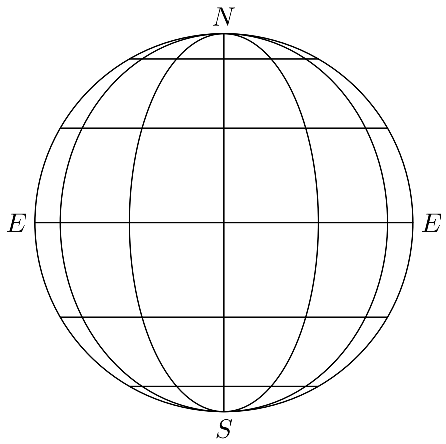
this factor is $u \cdot w$. Identifying $T_u \simeq \mathbf{R}^2$, $T_w \simeq \mathbf{R}^2$ the projection reads $p : \mathbf{R}^2 \to \mathbf{R}^2$, $p(\xi, \eta) = (x, y)$, $x = \xi$, $y = (u \cdot w)\eta$. The image of a circle $\xi^2 + \eta^2 = a^2$ in T_u is an *ellipse* in T_w with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{with } b = |u \cdot w|a, \qquad \varepsilon = \sqrt{1 - (u \cdot w)^2} \tag{5}$$

in which the *excentricity* also equals $\varepsilon = |u \times w| = |p_w(u)|$, as by (2) we have

$$|u \times w|^2 = (u \times w) \cdot (u \times w) = \begin{vmatrix} u \cdot u & u \cdot w \\ u \cdot w & w \cdot w \end{vmatrix} = 1 - (u \cdot w)^2 = |p_w(u)|^2$$

1.2. **Spherical aspects.** Let $\mathbb{S} \subset V$ be the *sphere* of unit vectors, e_1, e_2, e_3 an orthonormal basis and let $g : \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z} \to \mathbb{S}$, $g(\varphi, \lambda) = \cos\varphi \cos\lambda \cdot e_1 + \cos\varphi \sin\lambda \cdot e_2 + \sin\varphi \cdot e_3$ be the usual *geographical* coordinates. g is surjective, each point $u \in \mathbb{S}$ is determined by two angles: *latitude* φ and *longitude* λ. We have $g(\varphi + \pi, \lambda) = g(-\varphi, \lambda + \pi) = -g(\varphi, \lambda)$ (antipode). North pole is $g(+\pi/2, \lambda) = e_3$, South pole is $g(-\pi/2, \lambda) = -e_3$, the equatorial plane is $E = e_3^\perp = \mathbf{R}e_1 + \mathbf{R}e_2$.



We visualise this by means of the Earth and its grid. In the following figure we look at an image of the Earth with a 30° grid as seen from the equator. If you regard the Earth from a different point $z \notin E$ you obviously see one pole tilted towards you, the other being hidden on the back.

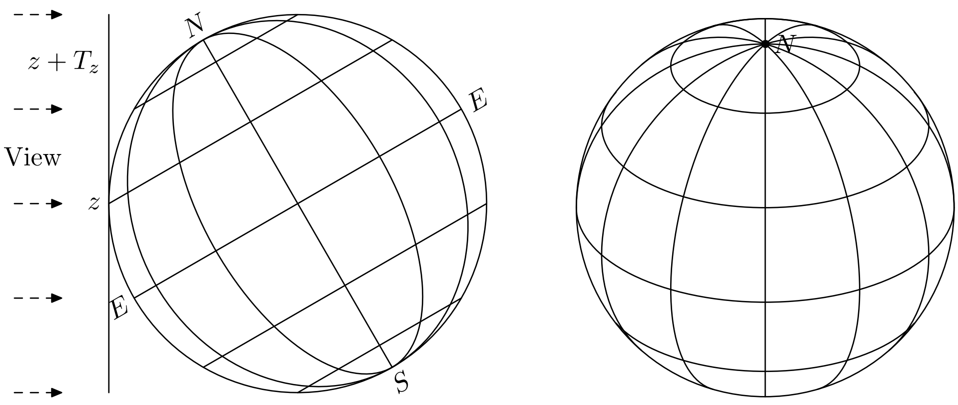
This raised ROEGEL's criticism, as this is seldomly accounted for in the scientific literature: he cites 10 celestial mechanical or astronomical works as well as websites of observatories and institutes on celestial mechanics and calculation of ephemerides which display objectionable representations.

For $z = g(\varphi, \lambda) \in \mathbb{S}$ let $z_1 = g(0, \lambda + \pi/2)$, $z_2 = g(\varphi + \pi/2, \lambda)$ and $T_z = z^\perp \subset V$ be the *orthogonal* plane to point z; then z, z_1, z_2 is an *orthonormal* basis of V. In particular we have $T_z = \mathbf{R}z_1 \oplus \mathbf{R}z_2$ and we take z_1, z_2 as basis vectors of the METAPOST-plane. The orthogonal projection $p_z : V \longrightarrow T_z$ is given by

$$p_z(v) = xz_1 + yz_2 \quad \text{with} \quad x = z_1 \cdot v,\ y = z_2 \cdot v. \tag{6}$$

ROEGEL in [4, 4.3] uses the following notation: his vectors $\vec{V}_1, \vec{V}_2, \vec{V}_3$ correspond to our z_1, z_2, z, but his angles ϕ, θ are $\phi = -\varphi$ and $\theta = \lambda + \pi/2$.

Some different view may assist our imagination. Let us tilt the Earth from the previous image to the left by $\pi/6$ (30°) and look at this configuration both from the direction of vector z_1 as well as its projection to the tangential plane T_z from the direction of vector z:



In the next sections we will first look at great circles and then generalize to arbitrary circles on spheres and determine their elliptical images in the tangential plane T_z.

1.3. **Great circles.** A great circle is an intersection of a plane with the sphere, hence of the form $K(u) = T_u \cap \mathbb{S}$ for a suitable vector $u \in \mathbb{S}$. If $u \cdot z < 0$ we replace u by $-u$, the circle $K(u)$ and the ellipse $p_z(K(u))$ do not change since $T_u = T_{-u}$. So, we may assume $u \cdot z \geq 0$.

Similarly to the orthonormal basis z, z_1, z_2 we choose an orthonormal basis u, u_1, u_2 of V. The equation of the ellipse $p_z(K(u)) \subset T_z$ is easy to establish. Let $v = \xi u_1 + \eta u_2 \in$

$K(u) \subset T_u$, by (6) we have

$$x = z_1 \cdot v = (z_1 \cdot u_1)\xi + (z_1 \cdot u_2)\eta$$
$$y = z_2 \cdot v = (z_2 \cdot u_1)\xi + (z_2 \cdot u_2)\eta$$

This system of linear equations in ξ, η yields by CRAMER's rule substituted into the circle equation $\xi^2 + \eta^2 = 1$ in T_u the equation of an ellipse in T_z

$$\begin{vmatrix} x & z_1 \cdot u_2 \\ y & z_2 \cdot u_2 \end{vmatrix}^2 + \begin{vmatrix} z_1 \cdot u_1 & x \\ z_2 \cdot u_1 & y \end{vmatrix}^2 = \begin{vmatrix} z_1 \cdot u_1 & z_1 \cdot u_2 \\ z_2 \cdot u_1 & z_2 \cdot u_2 \end{vmatrix}^2 . \tag{7}$$

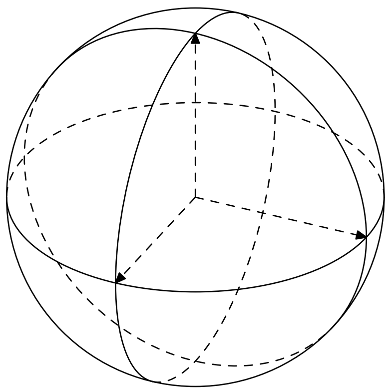
As is well-known the invariants of the ellipse can be calculated from the coefficients of this equation (see appendix A); this calculation is performed in section §A.2. Its simple result let me develop the conceptual approach I am presenting here.

In particular, we obtain for the image of the unit circle the equation (5) of an ellipse with $a = 1$ and $b = t \cdot u$. In case $b < 1$ the major axis of the ellipse has the line equation

$$(z_1 \cdot u)x + (z_2 \cdot u)y = 0 \tag{8}$$

Proof. If $b = z \cdot u < 1$, then $z \neq u$ and $T_z \neq T_u$. Hence the major axis is obviously the line $T_z \cap T_u$ and for a vector $v = xz_1 + yz_2 \in T_z$ we have $v \in T_u$ if and only if $v \cdot u = (z_1 \cdot u)x + (z_2 \cdot u)y = 0$. □

The two intersection points of the ellipse on the circumference $x^2 + y^2 = 1$ lie on the major axis. We need not calculate them as the equation (8) gives us the angle of the major axis and we can draw both half ellipses on the front and back side of the sphere.



As example serves a drawing of the sphere viewed from the direction $t = g(\varphi, \lambda)$ with coordinates $\varphi = 30°$, $\lambda = 25°$ and for vectors $u = e_1$, $u = e_2$ and $u = e_3$ (North pole) with the corresponding great circles $T_u \cap \mathbb{S}$.

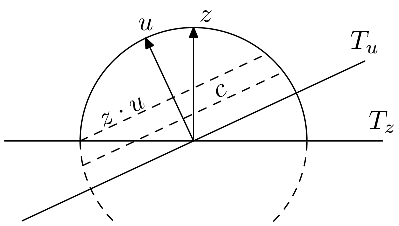
In this drawing I made the sphere *transparent* to show the complete ellipses including the hidden traces on the back side. The METAPOST code is listed in the appendix §C.1, the employed macro `circle` is explained in appendix §C.2.

1.4. **General circles.** We now treat arbitrary circles $K \subset \mathbb{S}$. As a planar figure K lies in a plane which is parallel to some linear plane T_u and is therefore an intersection $K = K(s,u) = (su + T_u) \cap \mathbb{S}$ with $-1 \le s \le +1$ if $K(s,u) \neq \emptyset$. We first deduce the invariants of the ellipse $p_z(K(s,u)) \subset T_z$, which is no more difficult than the case $s = 0$ that we have treated in the previous section. We exclude the degenerate case $p_z(u) = 0$.

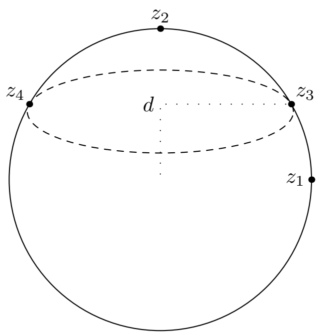
Let $v \in K(s,u)$, $v = su + w$ with $w \in T_u$ and set $c = |w|$, then $v \cdot v = s^2 + c^2 = 1$ and $w = p_u(v)$ runs in T_u on a circle of radius c.

For this reason $p_z(w)$ describes an ellipse in T_z with $a = c$, $b = |z \cdot u|c$ and major axis (8). Hence $p_z(v) = sp_z(u) + p_z(w)$ describes in T_z the ellipse shifted by $sp_z(u)$ with center $sp_z(u) = x_0 z_1 + y_0 z_2$ where $x_0 = s(z_1 \cdot u)$, $y_0 = s(z_2 \cdot u)$. Similarly, the major axis is shifted by the constant vector (x_0, y_0), hence its equation is $(z_1 \cdot u)(x - x_0) + (z_2 \cdot u)(y - y_0) = 0$.

When we look at a sphere, we can only see the front hemisphere $\mathbb{S}_z^{\ge}$ of points $v \in \mathbb{S}$ such that $z \cdot v \ge 0$, the back hemisphere $\mathbb{S}_z^{<}$ of v with $z \cdot v < 0$ is *invisible*. A general circle may be completely *visible* $K \subset \mathbb{S}_z^{\ge}$ or *invisible* $K \subset \mathbb{S}_z^{<}$, unlike a great circle. We ask for the exact conditions on $K(s,u)$ for being partly visible, and if so, we ask for the points where it crosses the border $\partial\mathbb{S}_z^{\geq} = T_z \cap \mathbb{S}$, that is we want to determine the intersection of the ellipse $p_z(K(s,u))$ with the circumference. The adjacent figure is a cross section through the sphere with the plane $\mathbf{R}z \oplus \mathbf{R}u$ and we can read off the condition from it: the ellipse $p_z(K(s,u))$ intersects the unit circle $x^2 + y^2 = 1$ in T_z if and only if $|z \cdot u| \leq c$ or equivalently $|s| \leq \varepsilon$. The intersection points $p_z(K(s,u)) \cap \mathbb{S} = \{z_3, z_4\}$ are osculation points of the ellipse with the surrounding circle. We can calculate z_3, z_4 from the circle and ellipse equations, but a conceptual approach is available.



For that purpose a parameter d with $-1 < d < +1$ is introduced and the following question asked:



Which ellipse of excentricity ε can be inscribed into the unit circle such that it touches the circle at the points $z_{3,4} = (\pm\sqrt{1-d^2}, d)$?

The ellipse with center $C = (0, y_0)$ (where $y_0 = s\varepsilon$) and excentricity ε has the equation

$$x^2 + \frac{(y-y_0)^2}{1-\varepsilon^2} = a^2$$

and at an osculation point its tangent has to equal the tangent of the circle, thus

$$\frac{d-y_0}{1-\varepsilon^2} = d$$

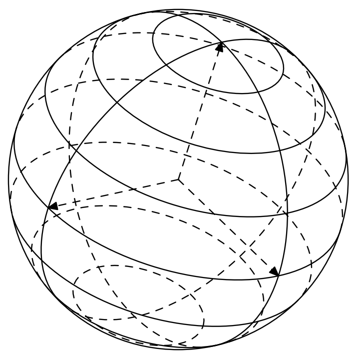
hence $y_0 = \varepsilon^2 d$ and consequently the connection to the parameter $s = \varepsilon d$, $d = s/\varepsilon$. Moreover its semi-major axis results to

$$a^2 = 1 - d^2 + \frac{(d-\varepsilon^2 d)^2}{1-\varepsilon^2} = 1 - d^2 + d^2(1-\varepsilon^2) = 1 - d^2\varepsilon^2 = 1 - s^2 = c^2$$

Now we interpret the drawing above invariantly under movement. To this end we view the METAPOST-plane as complex line $T_z \simeq \mathbf{C}$ and the rotations as multiplications by complex numbers ζ of absolute value $|\zeta| = 1$, i.e. $z_1 = 1$ und $z_2 = i$. The osculation points z_3, z_4 are symmetric to the unit vector $\zeta = p_z(u)/\varepsilon$. In the drawing above we had $\zeta = z_2 = i$, hence the general formula is $z_{3,4} = \zeta/i \cdot (\pm\sqrt{1-d^2}, d)$ or $z_4 = \zeta \cdot (d, \sqrt{1-d^2})$, $z_3 = \zeta \cdot (d, -\sqrt{1-d^2})$ and after complex multiplication (valid for $|s| \leq \varepsilon$)

(9) $$x_{3,4} = \frac{(z_1 \cdot u)s \pm (z_2 \cdot u)\sqrt{\varepsilon^2 - s^2}}{\varepsilon^2} \qquad y_{3,4} = \frac{(z_2 \cdot u)s \mp (z_1 \cdot u)\sqrt{\varepsilon^2 - s^2}}{\varepsilon^2}$$

The corresponding METAPOST definition for circles in general position is explained in appendix §C.2: the macro `circle` makes use of the formulas (9) to dissect the ellipses into two subpaths from z_3 to z_4 and from z_4 to z_3 and thus account for their visibility.



As example we take a similar drawing as for the great circles, this time with $\varphi = 15°$, $\lambda = 50°$ and as north pole $u = \sin\delta e_2 + \cos\delta e_3$, where the angle[1] is $\delta = 23.44°$ and draw parallels of latitude relative to u at $\psi = \pm 23.5°$, $\pm 45°$, $\pm 66.5°$. The parallels of latitude

[1] see §2.1 the *obliquity*

$p_z(K(\sin\psi, u))$ are drawn accounting for their visibility. We then have $s = \sin\psi$, $c = \cos\psi$ and the condition for intersection $|z \cdot u| \le c$ is equivalent to $|\sin\psi| \le \varepsilon$.

2. Application to Astronomy

The examples in section §1 visualised the globe with its geographical coordinate grid. Now we are going to devise and comprehend drawings common in astronomy. In this context the sphere $\mathbb{S} \subset V$ plays the role of the *celestial sphere* above us.

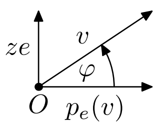
We explicate customary coordinate systems and *orbital elements* of celestial bodies like *satellite orbits* in the planetary system.

2.1. **Celestial coordinate systems.**

To determine the position of a celestial body a system of coordinates is necessary. To this end *spherical coordinates* are useful besides rectangular Cartesian coordinates. They are introduced as follows.

Let (E, e) be an orthogonal pair. In the plane E an orthonormal base e_1, e_2 is chosen: $E = \mathbf{R}e_1 \oplus \mathbf{R}e_2$. In this base a vector $v \in V$ has Cartesian coordinates $x = e_1 \cdot v$, $y = e_2 \cdot v$, $z = e \cdot v$, hence $v = xe_1 + ye_2 + ze$. Let $r = |v|$. In the plane E polar coordinates r_e, λ will be introduced

$$x = r_e \cos\lambda \qquad y = r_e \sin\lambda \qquad r_e = |p_e(v)| = \sqrt{x^2 + y^2}$$



Let $\varphi = \angle(p_e(v), v)$ be the angle between $p_e(v) \in E$ and v, hence $p_e(v) \cdot v = x^2 + y^2 = |p_e(v)||v|\cos\varphi$, thus $r_e = r\cos\varphi$, therefore

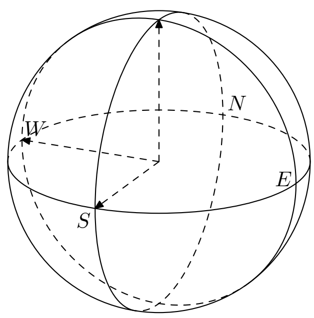
$$x = r\cos\varphi\cos\lambda \qquad y = r\cos\varphi\sin\lambda \qquad z = r\sin\varphi$$

$$v = xe_1 + ye_2 + ze = rg(\varphi, \lambda)$$

Astronomical coordinate systems are thus constructed like the geographical coordinate system. In astronomy different reference planes and axes are used, for instance

System	O	E	e	e_1	e_2	φ	λ
horizontal	♁	horizon	zenith	South	West	elevation h	azimuth A
equatorial	♁, ⊙	equator	North	♈	$e \times e_1$	declination δ	right asc. α
ecliptical	⊙	ecliptic	ENP	♈	$e \times e_1$	latitude β	longitude λ

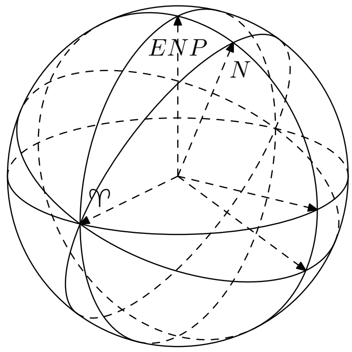
As the azimuth is counted towards West (in direction of the moving celestial sphere), the horizon system is a *left*-system, hence $e \times e_1 = -e_2$. This coordinate system is not an inertial system, but is rotating compared to the masses of the universe in one *sidereal* day, i.e. $23^h 56^m$. We show a picture of the horizontal system.



As to the other two systems we remark that neither the rotation axis of the Earth nor the ecliptic are really rigid but slowly shift in time.

The shift in time of the rotation axis is caused by the gravitational effects of the Sun and the Moon, producing a retrograd precession around the ecliptic pole in about 25,770 years. The influence of the planets on the orbit of the Earth causes an oscillation of the ecliptic about the mean position with a period of 41,000 years of about 0.85° (see [3, 1.4.1]). For the graphic representation these slow movements are irrelevant, though.

We consider the sun $\odot$ to be in the origin and the orthogonal pairs (E, e) and (E', e') constituted by the ecliptic E and the plane E' parallel to the equator of the Earth, both vectors e and e' point to their respective north poles. The angle[2] ε between E and E' (the same as between e and e') is the *obliquity* of the ecliptic (or *axial tilt*). Its current value at standard epoch J2000.0 is $\varepsilon = 23.44°$. The two planes intersect therefore in a line and $E \cap E' \cap \mathbb{S}$ contains two points: the *vernal equinox* $e_1 = \Upsilon$ and the *automnal equinox* $♎ = -e_1$.



By appendix §B.1 the base change in the plane $E_1 = e_1^{\perp}$ is $(e_2', e') = (e_2, e)\rho(-\varepsilon)$ and the *spherical transformation* formula reads $g(\delta, \alpha) = g(\beta, \lambda)\rho_1(-\varepsilon)$, see also [3, 1.3.3].

2.2. **Orbital elements.** We handle the task to describe the *orbits* of celestial bodies by *orbital elements* and draw the orbit on the celestial sphere with METAPOST. The *reference system* is chosen as the orthogonal pair (E, e) consisting of the *ecliptical* plane E and the ecliptical north pole (*ENP*) e. In addition we take e_1 as above pointing to the *vernal equinox* Υ and $e_2 = e \times e_1$. The rectangular coordinates refer to this base. By choice we take as origin $O = \odot$ the sun, or in case of terrestrial satellites $O = ♁$ the Earth.

The considered heavenly body moves in a plane T_u with u as unit vector of the *orbital angular momentum*. Let the *inclination* angle of E and T_u be i, i.e. $e \cdot u = \cos i$. We have $e \cdot u \geq 0$ for $0 \leq i \leq \pi/2$ and $e \cdot u < 0$ (*retrograde* revolution) for $\pi/2 < i < \pi$. As by definition i is the *pole distance* of u we have $\beta(u) = \pi/2 - i$.

The line $E \cap T_u$ is called *line of nodes*, the two points $E \cap T_u \cap \mathbb{S} = \{☊, ☋\}$ are the *ascending* and *descending* node of the orbit. The ecliptical longitude $\Omega = \lambda(☊)$ is the *longitude of the ascending node*. From this it follows that $\lambda(u) = \Omega - \pi/2$ and hence the ecliptical coordinates of u are

$$u = g(\beta(u), \lambda(u)) = g(\pi/2 - i, \Omega - \pi/2) = (\sin\Omega \sin i, -\cos\Omega \sin i, \cos i)$$

In the orbit plane T_u we choose u_1, u_2 such that u_1 points to the *periapsis*. Let ω be the *longitude of the periapsis* from the ascending node measured in the orbital plane.

By appendix §B.1 we have for the base change

$$(u_1, u_2, u) = (e_1, e_2, e)\rho_3(\Omega)\rho_1(i)\rho_3(\omega),$$

hence for a vector $v = xe_1 + ye_2 + ze = x'u_1 + y'u_2 + z'u$ the coordinate transformation rule is

$$(x', y', z') = (x, y, z)\rho_3(\Omega)\rho_1(i)\rho_3(\omega).$$

Therefore the ecliptical coordinates are $(x, y, z) = (x', y', z')\rho_3(-\omega)\rho_1(-i)\rho_3(-\Omega)$ and we have

$$u_1 = (\cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega, \cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega, \sin\omega\sin i)$$
$$u_2 = (-\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega, -\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega, \cos\omega\sin i)$$
$$u = (\sin i\sin\Omega, -\sin i\cos\Omega, \cos i)$$

u_1, u_2 are the GAUSS *vectors* of [3, 1.3.1]: $u_1 = P$, $u_2 = Q$.

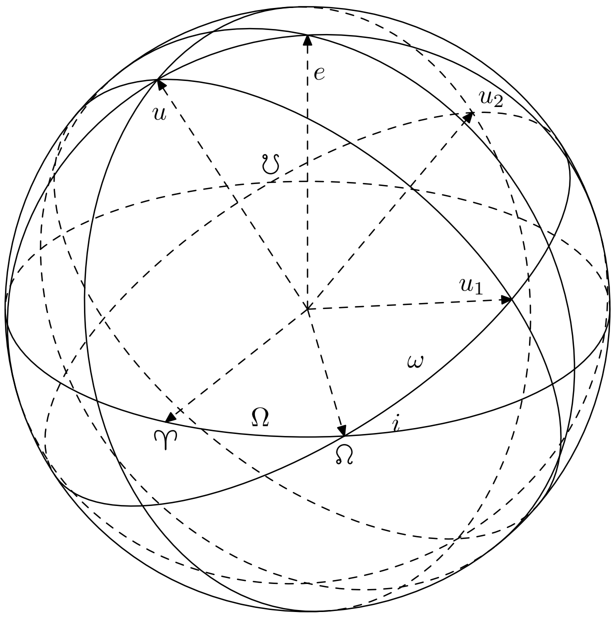
When writing $u_1 = G(\omega, i, \Omega)$ as function of ω, i, Ω, then we have

$$u_2 = G(\omega + \pi/2, i, \Omega) \qquad u = G(\pi/2, i + \pi/2, \Omega) = g(\pi/2 - i, \Omega - \pi/2)$$

and $G(v + \omega, i, \Omega) = \cos(v)u_1 + \sin(v)u_2$.

[2] the excentricity ε will not occur in section §2.1

Here is a drawing with the values $i = 30°$, $\Omega = 35°$, $\omega = 43°$ viewed from the perspective of $z = g(\varphi, \lambda)$ with $\varphi = 25°$, $\lambda = 28°$. Of course these values are arbitrary, in the appendix §C.3 the METAPOST-code is documented.



2.3. **Orbit projection.** After having explained the *orbit elements* we want to draw the *orbits of celestial bodies* themselves by orthogonal projections. In general, these orbits are not circular, but according to the gravitational law *planar curves of second order* (at least approximately, up to perturbations). The difference to the previous section §2.2 is thus, that we are looking for a drawing with METAPOST of $p_z(K)$ where $K \subset T_u$ is the *real* orbit in space.

In §1.2 we distinguished a basis in the METAPOST-plane $T_z = \mathbf{R}z_1 \oplus \mathbf{R}z_2$. In §2.2 we determined as well a basis in the orbit plane $T_u = \mathbf{R}u_1 \oplus \mathbf{R}u_2$. Therefore we get isomorphisms

$$\begin{aligned} \mathbf{R}^2 &\xrightarrow{\sim} T_u \\ (\xi, \eta) &\longmapsto \xi u_1 + \eta u_2 \end{aligned} \qquad\qquad \begin{aligned} \mathbf{R}^2 &\xrightarrow{\sim} T_z \\ (x, y) &\longmapsto x z_1 + y z_2 \end{aligned}$$

The orthogonal projection p_z restricted to the orbit plane induces hence a *transition matrix* $m(p_z|T_u)$ in the following diagram

$$\begin{array}{ccc} \mathbf{R}^2 & \xrightarrow{\sim} & T_u \\ {\scriptstyle m(p_z|T_u)}\downarrow & & \downarrow{\scriptstyle p_z|T_u} \\ \mathbf{R}^2 & \xrightarrow{\sim} & T_z \end{array}$$

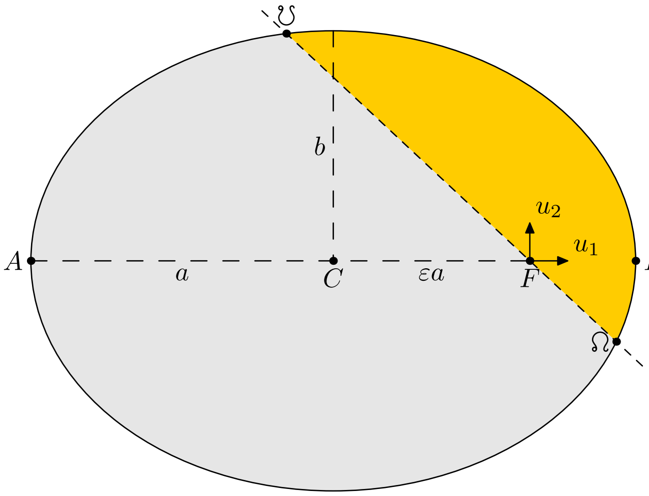
such that for all $(\xi, \eta), (x, y) \in \mathbf{R}^2$ with $p_z(\xi u_1 + \eta u_2) = x z_1 + y z_2$ the relation $(x, y) = (\xi, \eta) \cdot m(p_z|T_u)$ is satisfied.

We apply the calculation of the *transition matrix* in §B.2 to $E = T_u$, $E' = T_z$ and the transformation $t = p_z|T_u$. For $w \perp z$ we have $p_z(v) \cdot w = v \cdot w$, hence by (15) the transition matrix is

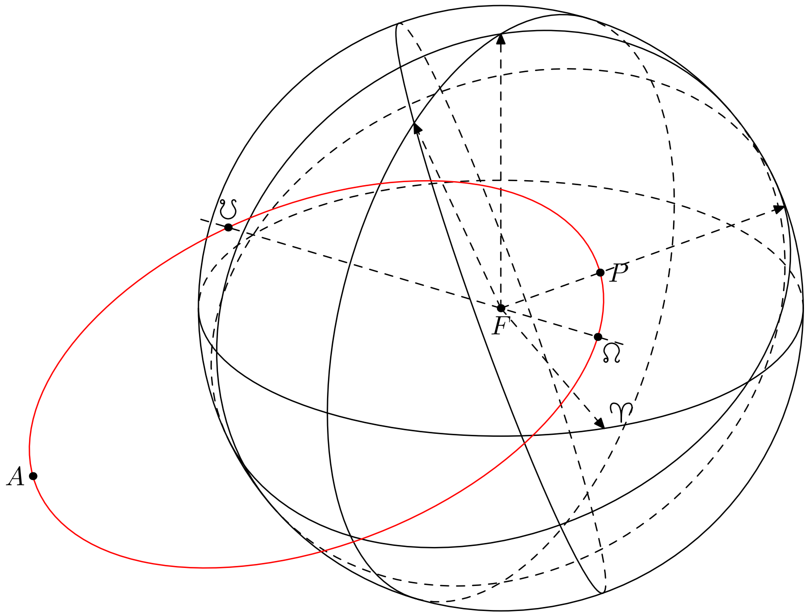
$$m(p_z|T_u) = \begin{pmatrix} u_1 \cdot z_1 & u_1 \cdot z_2 \\ u_2 \cdot z_1 & u_2 \cdot z_2 \end{pmatrix}.$$

Its determinant is $\det m(p_z|T_u) = (u_1 \times u_2) \cdot (z_1 \times z_2) = u \cdot z$ by (2).

Let us look at an orbit in its orbit plane T_u. Let E be the ecliptic and e the unit vector to the ecliptical north pole. The focus F is in the origin, the vector u_1 points at the periapsis P (A is the apoapsis). The ascending node ☊ has the angle ω with u_1 and the descending node ☋ the angle $\pi - \omega$. Recall by (8) that the line of nodes $T_e \cap T_u$ is given in T_u by the equation $(u_1 \cdot e)\xi + (u_2 \cdot e)\eta = 0$. The yellow part is above the ecliptic, the grey one below.



When we apply the tranformation $t = p_z | T_u$ to the orbit we obtain



Appendix A. Equations of ellipses and their invariants

The invariants of plane curves of second order (like *ellipses*) follow readily from the coefficients of the equation of the curve. This applies in particular to major and minor semi-axes a and b, as well as to the line equations of major and minor axis. The knowledge of these invariants immensely facilitates the METAPOST-drawing. The relevant formulas are collected here, for details see the article [5].

A.1. **Invariants of curves of second order.** A plane curve of second order ist given by an equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \tag{10}$$

where the coefficients a_{ik} are subject to a common factor. The *characteristic* polynomial of the curve is

$$\begin{vmatrix} a_{11}-X & a_{12} \\ a_{12} & a_{22}-X \end{vmatrix} = X^2 - t_{33}X + A_{33} = (X-\lambda_1)(X-\lambda_2)$$

where $t_{33} = a_{11}+a_{22}$, $A_{33} = a_{11}a_{22}-a_{12}^2$. Its roots are the *eigenvalues* λ_1, λ_2. They are determined up to an arbitrary factor: if you multiply (10) by q the new eigenvalues become $q\cdot\lambda_1$, $q\cdot\lambda_2$. The sign of the coefficients will be chosen such that $t_{33} \le 0$ and the eigenvalues will be numerated such that $\lambda_1 \ge \lambda_2$. We are concerned with the *elliptic* case $A_{33} > 0$ only. We quote the formulas for the semi-axes

(11) $$a = -\frac{1}{\lambda_1}\sqrt{\frac{D}{-\lambda_2}}, \quad b = -\frac{1}{\lambda_2}\sqrt{\frac{D}{-\lambda_1}}$$ [5, (2.14-15)]

and the line equation for the *major axis*

(12) $$(a_{11}-\lambda_1)(x-x_0) + a_{12}(y-y_0) = 0$$ [5, (2.46)]

Here $D = \det(a_{ik})$ is the determinant of the coefficients and $C = (x_0, y_0)$ is the center of the curve:

(13) $$x_0 = \frac{A_{13}}{A_{33}}, \quad y_0 = \frac{A_{23}}{A_{33}}$$ [5, (2.32)]

A.2. **Calculation of invariants of ellipse equation.** We apply the theory above to the ellipse equation (7) in section §1.3. With the conventions of §A.1 its coefficients are

$$\begin{aligned} a_{11} &= -(z_2\cdot u_1)^2 - (z_2\cdot u_2)^2 & a_{12} &= (z_1\cdot u_1)(z_2\cdot u_1) + (z_1\cdot u_2)(z_2\cdot u_2) \\ a_{22} &= -(z_1\cdot u_1)^2 - (z_1\cdot u_2)^2 & a_{33} &= ((z_1\cdot u_1)(z_2\cdot u_2) - (z_1\cdot u_2)(z_2\cdot u_1))^2 \\ a_{13} &= 0 & a_{23} &= 0 \end{aligned}$$

This implies already $A_{13} = A_{23} = 0$ and the center of the ellipse by (13) is the origin $C = (0,0)$.

We will simplify the other coefficients by exploiting the fact that z, z_1, z_2 and u, u_1, u_2 are orthonormal bases. For example we have $v = (v\cdot u)u + (v\cdot u_1)u_1 + (v\cdot u_2)u_2$ for any $v \in V$. This yields for $v = z_1, z_2$ because of $z_1\cdot z_2 = 0$ the relation $(z_1\cdot u)(z_2\cdot u) + (z_1\cdot u_1)(z_2\cdot u_1) + (z_1\cdot u_2)(z_2\cdot u_2) = 0$, hence $a_{12} = -(z_1\cdot u)(z_2\cdot u)$. Similarly ensues the relation $z_1\cdot z_1 = (z_1\cdot u)^2 + (z_1\cdot u_1)^2 + (z_1\cdot u_2)^2 = 1$, hence $a_{22} = (z_1\cdot u)^2 - 1$, and in the same manner we obtain $a_{11} = (z_2\cdot u)^2 - 1$. By using $z = \pm z_1\times z_2$, $u = \pm u_1\times u_2$ we have by (2) $z\cdot u = \pm(z_1\times z_2)\cdot(u_1\times u_2) = \pm\begin{vmatrix} z_1\cdot u_1 & z_1\cdot u_2 \\ z_2\cdot u_1 & z_2\cdot u_2 \end{vmatrix}$, hence $a_{33} = (z\cdot u)^2$.

Summing up, the coefficients of (7) are

$$\begin{aligned} a_{11} &= (z_2\cdot u)^2 - 1 & a_{22} &= (z_1\cdot u)^2 - 1 & a_{33} &= (z\cdot u)^2 \\ a_{12} &= -(z_1\cdot u)(z_2\cdot u) & a_{13} &= 0 & a_{23} &= 0 \end{aligned}$$

As $u = (z\cdot u)z + (z_1\cdot u)z_1 + (z_2\cdot u)z_2$ we have $1 = (z\cdot u)^2 + (z_1\cdot u)^2 + (z_2\cdot u)^2$, thus $t_{33} = a_{11} + a_{22} = -(z\cdot u)^2 - 1$. Moreover $A_{33} = a_{11}a_{22} - a_{12}^2 = -(z_1\cdot u)^2 - (z_2\cdot u)^2 + 1 = (z\cdot u)^2$. For the eigenvalues λ_1, λ_2 we thus obtain $\lambda_1 + \lambda_2 = -(z\cdot u)^2 - 1$ and $\lambda_1\lambda_2 = (z\cdot u)^2$, hence $\lambda_1 = -(z\cdot u)^2$, $\lambda_2 = -1$.

By (11) and $D = a_{33}A_{33} = (z\cdot u)^4$ follows $a = 1$, $b = z\cdot u$, since we assumed $z\cdot u \ge 0$ in section §1.3.

We have $a_{11} - \lambda_1 = (z_2\cdot u)^2 - 1 + (z\cdot u)^2 = -(z_1\cdot u)^2$. The equation of the major axis by (12) is $-(z_1\cdot u)^2 x - (z_1\cdot u)(z_2\cdot u)y = 0$, which for $z_1\cdot u \ne 0$ can be simplified to $(z_1\cdot u)x + (z_2\cdot u)y = 0$. Remark: this line equation is also valid for $z_1\cdot u = 0$.

A.3. **Equation and invariants in general position.** We do not need the equation for the ellipse $p_z(K(s,u)) \subset T_z$, but it is easy enough to write down.

The same approach as in §1.3 leads to the equation

$$\begin{vmatrix} x-x_0 & z_1\cdot u_2 \\ y-y_0 & z_2\cdot u_2 \end{vmatrix}^2 + \begin{vmatrix} z_1\cdot u_1 & x-x_0 \\ z_2\cdot u_1 & y-y_0 \end{vmatrix}^2 = \begin{vmatrix} z_1\cdot u_1 & z_1\cdot u_2 \\ z_2\cdot u_1 & z_2\cdot u_2 \end{vmatrix}^2 \cdot c^2$$

which can be treated as in the previous section §A.2. This yields an equation (10) with the coefficient matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} (z_2\cdot u)^2-1 & -(z_1\cdot u)(z_2\cdot u) & s(z_1\cdot u) \\ -(z_1\cdot u)(z_2\cdot u) & (z_1\cdot u)^2-1 & s(z_2\cdot u) \\ s(z_1\cdot u) & s(z_2\cdot u) & (z\cdot u)^2-s^2 \end{pmatrix}$$

which has the same eigenvalues as in §A.2, in particular $A_{33}=(z\cdot u)^2$. The cofactors A_{13} and A_{23} are

$$A_{13} = \begin{vmatrix} -(z_1\cdot u)(z_2\cdot u) & (z_1\cdot u)^2-1 \\ s(z_1\cdot u) & s(z_2\cdot u) \end{vmatrix} = s(z_1\cdot u)\begin{vmatrix} -(z_2\cdot u) & (z_1\cdot u)^2-1 \\ 1 & (z_2\cdot u) \end{vmatrix} = s(z_1\cdot u)(z\cdot u)^2$$

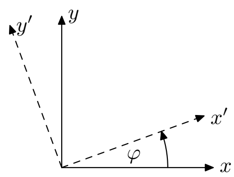
$$A_{23} = \begin{vmatrix} -(z_1\cdot u)(z_2\cdot u) & (z_2\cdot u)^2-1 \\ s(z_2\cdot u) & s(z_1\cdot u) \end{vmatrix} = s(z_2\cdot u)\begin{vmatrix} -(z_1\cdot u) & (z_2\cdot u)^2-1 \\ 1 & (z_1\cdot u) \end{vmatrix} = s(z_2\cdot u)(z\cdot u)^2$$

The determinant D is

$$\begin{aligned} D &= a_{13}A_{13}+a_{23}A_{23}+a_{33}A_{33} = (z\cdot u)^2\big(s^2(z_1\cdot u)^2+s^2(z_2\cdot u)^2+(z\cdot u)^2-s^2\big) = \\ &= (z\cdot u)^4(1-s^2) = (z\cdot u)^4c^2. \end{aligned}$$

Appendix B. Base change

B.1. **Rotation matrices.** In this section we introduce *rotation matrices* $\rho(\varphi)$. In a Euclidean plane E we consider two orthonormal bases (e_1,e_2), (e'_1,e'_2) and their respective coordinate systems (x,y), (x',y') in which a vector $v\in E$ has the equation $v = xe_1+ye_2 = x'e'_1+y'e'_2$, and let the primed system be generated from the unprimed system by a *rotation* of angle φ. Then the following relations hold



$$\begin{aligned} x' &= x\cos\varphi + y\sin\varphi \\ y' &= -x\sin\varphi + y\cos\varphi \end{aligned}$$

Let the *rotation matrix* of the angle φ be defined by

$$\rho(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

then we can write the *transformation rules*

$$\begin{aligned} (x',y') &= (x,y)\cdot\rho(\varphi) \\ (e'_1,e'_2) &= (e_1,e_2)\cdot\rho(\varphi) \end{aligned}$$

Similarly we define *rotation matrices* in space $V=\mathbf{R}e_1\oplus\mathbf{R}e_2\oplus\mathbf{R}e_3$

$$\rho_1(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \qquad \rho_2(\varphi) = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

$$\rho_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which respectively represent rotation around the x-, y- and z-axis. Rotation matrices satisfy the equation $\rho(-\varphi) = {}^t\rho(\varphi) = \rho(\varphi)^{-1}$.

B.2. **Transition matrix.** We consider two planes E,E' with given base vectors e_1,e_2 resp. e'_1,e'_2. A linear transformation $t: E\to E'$ is determined by the images $t(e_1)$ and $t(e_2)$

$$\begin{aligned} t(e_1) &= ae'_1 + be'_2 \\ t(e_2) &= ce'_1 + de'_2 \end{aligned}$$

In the diagram

$$\begin{array}{ccccc} \mathbf{R}^2 & \xrightarrow{\sim} & E & \qquad & (x,y) \mapsto xe_1 + ye_2 \\ {\scriptstyle m(t)}\downarrow & & \downarrow{\scriptstyle t} & & \\ \mathbf{R}^2 & \xrightarrow{\sim} & E' & & (x',y') \mapsto x'e_1' + y'e_2' \end{array} \tag{14}$$

the matrix $m(t)$ of the linear transformation t is given by

$$m(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If the basis e_1', e_2' is orthonormal, we have

$$m(t) = \begin{pmatrix} t(e_1)\cdot e_1' & t(e_1)\cdot e_2' \\ t(e_2)\cdot e_1' & t(e_2)\cdot e_2' \end{pmatrix} \tag{15}$$

In the orbit projection in §2.3 we apply to the left hand side of (14) the METAFONT-transformation $t = (0,0,a,c,b,d)$ [2, chap. 15]

$$(x,y)\ \texttt{transformed}\ \ t = (ax+cy, bx+dy) = (x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Appendix C. METAPOST Code

My first calculations that led to this article have been done in April 2010, after acquiring in February the new edition of *The LaTeX Graphics Companion* [1] and admiring in particular the drawings by Roegel ([1, lunar orbit, page xx], [4]). These calculations aimed at making the rather smart METAPOST constructions by Roegel unnecessary and replace them by mathematical concepts.

C.1. **Transparent sphere with great circles.** The definition of $\texttt{halfell}(a,b,\theta)$ describes half an ellipse with center $(0,0)$ and semi-axes a, b, which has been rotated by θ. By the line equation (8) of the major axis $(z_1\cdot u)x + (z_2\cdot u)y = 0$ the angle results to $\theta = \texttt{angle}(-(z_2\cdot u),(z_1\cdot u))$.

In section 1.2 we introduced *spherical coordinates*

$$\begin{aligned} z &= g(\varphi,\lambda) = (\cos\varphi\cos\lambda, \cos\varphi\sin\lambda, \sin\varphi) \\ z_1 &= g(0,\lambda+\pi/2) = (-\sin\lambda, \cos\lambda, 0) \\ z_2 &= g(\varphi+\pi/2,\lambda) = (-\sin\varphi\cos\lambda, -\sin\varphi\sin\lambda, \cos\varphi) \end{aligned}$$

These are used in the METAPOST code in the macro `circle` as function `g(phi,lambda)`. `circle` is explained in §C.2. The essential part here is

```
p1 = halfell(a,c0*a,angle(-c2,c1)); % visible path
p2 = halfell(a,c0*a,angle(c2,-c1)); % invisible path
```

The METAPOST-code for the drawing is as follows

```
%  ------------------------------
%  Figure 5 -- great circles, parametrized
%  ------------------------------
beginfig(5);
phi:=30;                                % latitude
lambda:=25;                             % longitude
vector e[];
e1=(1,0,0);
e2=(0,1,0);
e3=(0,0,1);
draw fullcircle scaled 2a;
circle(e1,0,a,3);
circle(e2,0,a,3);
circle(e3,0,a,3);
endfig;
```

C.2. **Circles in general position.** The METAPOST code is in fact very simple:

```
% ------------------------------
% Figure 8 -- circles of latitude
% ------------------------------
beginfig(8);
phi:=15;                                % latitude
lambda:=50;                             % longitude
vector e[];
e1=(1,0,0);                             % vector e1, pointing to vernal equinox
e2=(0,cosd(del),-sind(del));            % vector e2, axial tilt
e3=(0,sind(del),cosd(del));             % vector e3 as North pole, axial tilt
draw fullcircle scaled 2a;
circle(e1,0,a,3);
circle(e2,0,a,3);
circle(e3,0,a,3);
for psi:= 23.5,45,66.5, -23.5,-45,-66.5:
    circle(e3,psi,a,2);                 % circles of latitude psi
endfor;
endfig;
```

The parameter s is chosen now as $s = \sin\psi$ with $-\pi/2 < \psi < +\pi/2$, $c = \cos\psi$. A circle in general position $K(\sin\psi, u)$ is projected by p_z into an ellipse, which is drawn by the macro `circle`(u, ψ, a, f) in the METAPOST-plane T_z.

In the case of great circles ($\psi = 0$) we use the simplification from section C.1. In case of intersection points of $p_z(K(s,u))$ with $x^2 + y^2 = 1$ we use the formulas (9) for the definition of the vectors z_3, z_4. We form the intersection of the lines from the center of the ellipse to the intersection points z_3, z_4, take their *time* parameters (`intersectiontimes`) c_3, c_4 and draw *continuously* the *visible* part of the ellipse corresponding to the arc $K(s,u) \cap \mathbb{S}_z^{\geqq}$ on the visible hemisphere (`path` p_1) and draw the *invisible* part which matches the invisible arc $K(s,u) \cap \mathbb{S}_z^{\leqq}$ in *dashed* line (`path` p_2). You will notice some fine points in the code concerning the visibility of the ellipses in case of non-intersection.

The parameter a serves for scaling (size unit). The parameter f serves as flag: $f = 0$ draws only the visible path p_1, $f = 1$ draws also the vector $p_z(u)$ (path p_0), $f = 2$ draws the invisible path p_2 (dashed evenly) and $f = 3$ draws both.

```
% -------------------------------------------------------------------
%   circle parms
%   u: input vector u, to be projected via z=g(phi,lambda)
%   psi: latitude relative to vector u in range -90 < psi < + 90
%   a: size, scale unit
%   f: flag f=0,1,2,3
%           f=0 draws only the visible path p1
%           f=2, 3 draws the invisible path p2
%           f=1, 3 draws vector path p0 to p_z(u)
% -------------------------------------------------------------------
def circle(expr u,psi,a,f)=
    numeric s, c, e, c[]; path p[]; pair q[];

    c0=dot(u,g(phi,lambda));
    c1=dot(u,g(0,lambda+90));
    c2=dot(u,g(phi+90,lambda));

    e  = 1+-+c0;                        % epsilon excentricity
    s  = sind(psi); c  = cosd(psi);     %
    q1 = (a,0); q2 = (0,a);             % base vectors of METAPOST plane
    w$:=q0=c1*q1 + c2*q2;               % p_z(u), saved in global variable w$
```

```
    p0 = origin--q0;                    % vector arrow
    p9 = ellipse(c*a,abs(c0)*c*a,angle(c2,-c1)) shifted (s*q0);

    if (psi=0):                         % great circle
        p1 = halfell(a,c0*a,angle(-c2,c1)); % visible path
        p2 = halfell(a,c0*a,angle(c2,-c1)); % invisible path
    elseif (abs(c0)>c):                 % circles of latitude psi, no intersection
        if (s*c0>0):                    % visibility of complete ellipse
            p1=p9;
            p2=origin;
        else:                           % invisibility of complete ellipse
            p1=origin;
            p2=p9;
        fi
    else:                               % intersection |s|<= e, circle of latitude psi
        c5=e+-+s; c6=e*e;
        q3=((c1*s+c2*c5)*a/c6,(c2*s-c1*c5)*a/c6);
        q4=((c1*s-c2*c5)*a/c6,(c2*s+c1*c5)*a/c6);
        c3=xpart(p9 intersectiontimes ((s*q0)--1.1q3));
        c4=xpart(p9 intersectiontimes ((s*q0)--1.1q4));
        if (psi>0):
            p3 = subpath(c4,8) of p9 & subpath(0,c3) of p9;
            p4 = subpath(c3,c4) of p9;
        else:
            p3 = subpath(c4,c3) of p9;
            p4 = subpath(c3,8) of p9 & subpath(0,c4) of p9;
        fi

        if (c0>0):
            p1=p3; p2=p4;
        else:                           % if (z.u)<0 we interchange visible and invisible
            p1=p4; p2=p3;
        fi
    fi

    draw p1;
    if (f>1):
        draw p2 dashed evenly;
    fi
    if (f=1) or (f=3):
        drawarrow p0 dashed evenly;
    fi

enddef;
```

C.3. **Orbital elements.** To be able to mark the vectors $p_z(u)$ with labels the coordinates (in the METAPOST-plane) that have been calculated inside the macro `circle` are stored in the global variable `w$`. The GAUSS vector $u_1 = G(\omega, i, \Omega)$ is denoted by the function `G(omega,incl,Omega)`. In other respects the code is straightforward.

```
%  ------------------------------
%  Figure 12 -- orbital elements
%  ------------------------------
beginfig(12);

phi:=25;                                % latitude
lambda:=28;                             % longitude
```

```
Omega=35;                               % longitude of ascending node
incl=30;                                % inclination
omega=43;                               % longitude of periapsis in orbital plane
vector e[],u[];
e1=(1,0,0);                             % aries - vernal equinox
e2=(0,1,0);                             % e2=e x e1 - not used
e3=(0,0,1);                             % e3=e=ENP
u0=g(0,Omega);                          % ascending node
u1=G(omega,incl,Omega);                 % Gauss vector 1, pointing to periapsis
u2=G(omega+90,incl,Omega);              % Gauss vector 2
u3=g(90-incl,Omega-90);                 % orbital angular momentum, vector u

draw fullcircle scaled 2l;              % circumference
circle(e1,0,1,3);                       % vernal equinox e1, circle thru e and e2
label.bot(aries,w$);
z1=w$;
circle(e3,0,1,3);                       % ecliptic thru e1 and e2
label.lrt(btex $e$ etex,.9w$);          % ecliptical north pole e
circle(u0,0,1,3);                       % ascending node, circle thru e and u
z0=w$;
label.bot(ascnode,z0);                  % ascending node
label.top(descnode,-z0);                % descending node
circle(u3,0,1,3);                       % orbit in T_u
label.llft(btex $u$ etex,0.9w$);        % orbital angular momentum vector u
circle(u1,0,1,3);                       % circle thru u and u2
z1'=w$;                                 % z1'=p_z(u1)
label.ulft(btex $u_1$ etex,.9z1');
circle(u2,0,1,3);                       % circle thru u and u1
label.urt(btex $u_2$ etex,w$);          % u2

label(btex $\Omega$ etex,.5[.9z1,.9z0]);
label(btex $\omega$ etex,.5[.9z0,.9z1']);
label.lrt(btex $i$ etex,.2[z0,1.2z1']);

endfig;
```

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