NOTES TO THE 'ESSAYS IN THE HISTORY OF LIE GROUPS AND ALGEBRAIC GROUPS' BY ARMAND BOREL

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ABSTRACT. Notes taken during my reading of the "Essays in the History of Lie Groups and Algebraic Groups" by Armand Borel, with some corrections to misprinted formulae.

PREFACE

This short note was written during the two weeks from August 27 to September 8, 2001, when I was taking notes during my reading of Borel's excellent book 'Essays in the History of Lie Groups and Algebraic Groups' [1].

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1. CHAPTER 2

Something must have mixed up the references. When Borel talks about a *paragraph* (§) he probably means a section (maybe, the TEX formatting process introduced the §§, and the *now* section numbers were his former §§ – who knows ?).

Here is how they relate to each other:

- §1 consists of sections 1–3
- §2 consists of sections 4–7
- §3 consists of sections 8–11
- §4 consists of sections 12–14

§3. Algebraic proofs of full reducibility. The title should *not* read "Algebraic peoofs of full reducibility", but rather as indicated here ! It is strange that this 'misprint' isn't reflected in the table of contents !

In section 8 Borel discusses the contribution of H. Casimir studying the representations of the rotation group $\mathbf{SO}_3$ in $\mathbf{R}^3$. Its Lie algebra $\mathfrak{so}_3$ is generated by the infinitesimal rotations D_x, D_y, D_z around the three coordinate axes in $\mathbf{R}^3$ – and that's where my trouble started, as I got a different matrix representation from that in Borel's text.

There is an obvious misprint in formula (11), where $D_z = y.\partial_y - x.\partial_x$ should read $D_z = y.\partial_x - x.\partial_y$ (— or rather $D_z = x.\partial_y - y.\partial_x$? — We will see in a moment that the first is correct). I verified the obvious requirements (like $[D_x, D_y] = D_z$) and at a closer look found a disturbing sign difference with what I would have written: my D_x, D_y are the negative from those of Borel (but the same D_z !). I also expected that – by symmetry – the alternate group would operate on the coordinates and switch $x \to y \to z \to x$.

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I could not easily figure out where this change of sign comes from. Therefore I decided to sit down and write it out in all gory detail (below).

An additional misinterpretation that I committed during the process of clarification was that I mixed up $V = \mathbf{R}^3$ with the dual V^*, and as the differential operators are expressed in terms of the local variables x, y, z (which are the linear functionals $V \to \mathbf{R}$ corresponding to the coordinate projections), my confusion was complete.

Resolution of my confusion. Let $x = pr_1$, $y = pr_2$, $z = pr_3$ denote the coordinate projections $V = \mathbf{R}^3 \longrightarrow \mathbf{R}$. Borel denotes the polynomial algebra on V with $\mathbf{R}[V]$, which is nothing but $\mathbf{R}[V] = \mathbf{R}[x, y, z]$ and can be identified with the symmetric algebra $\mathbf{S}^\bullet(V^*)$ on the *dual* of V. In particular the *linear* elements (of degree 1) are $\mathbf{S}^1(V^*) = \mathbf{R}x + \mathbf{R}y + \mathbf{R}z = V^*$, the dual vector space to V.

Let's consider for an instant the general situation, where

$$\pi : G \longrightarrow GL(V)$$

is a finite dimensional representation and

$$d\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

is the derived representation of the Lie algebra $\mathfrak{g}$ of G. It is defined by

$$d\pi(X)v := \frac{d}{dt}\pi(exp(t.X))v|_{t=0}$$

Now we consider the operation of the group on the functions $f : V \longrightarrow \mathbf{R}$ by transport of structure, i.e.

$$\rho(x)f(v) = f(\pi(x^{-1})v)$$

This, in particular, gives rise to an operation of its Lie algebra on $\mathbf{R}[V]$ that is usually expressed in terms of differential operators. The result of the differentiation for a linear element $f \in V^* = \mathbf{S}^1(V^*)$ is then

$$d\rho(X)f = f \circ d\pi(-X)$$

We will apply this to the situation $G = \mathbf{SO}_3$, π is the inclusion $G \subset GL(V)$ for $V = \mathbf{R}^3$ (the 'identity' or *standard* representation). The rotation around the z-axis is given by

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp t.Z$$

where the matrix Z is

$$Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $Z = D_z$ in Borel's notation.

Let us denote the dual basis in V to x, y, z in V^* by e_x, e_y, e_z, then

$$Ze_x = e_y, \quad Ze_y = -e_x, \quad Ze_z = 0$$

and on the dual space we get

$$x \circ Z = -y, \quad y \circ Z = x, \quad z \circ Z = 0$$

or otherwise put (we make use of $Zf = -f \circ Z$ for linear f, see above)

$$Z = y.\partial_x - x.\partial_y$$

(it suffices to test this on the basis, $f = x, y, z$).

Now, I apply the cyclic permutation $x \to y \to z \to x$ and get the corresponding rotations around x- and y-axis:

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and as operator on the polynomial algebra $\mathbf{R}[V]$ we have

$$X = z.\partial_y - y.\partial_z, \qquad Y = x.\partial_z - z.\partial_x$$

We see that $X = -D_x$ and $Y = -D_y$ in Borel's book (i.e. he takes the positive rotation around the z-axis, but the negative around the x- and y-axis).

So, all the formulas (except the misprint for D_z) *are* correct in Borel's book – although two of the rotations go the other way round from what one would expect (at least me).

2. Chapter 3

On page 31 (near the end of §1) there is an extraneous 'be' in the sentence:

"This can ~~be~~ already be seen by the rather sharply worded footnote in his answer to Study, ... "

On page 35 (§2, section 4), formula (5) should read

$$A(\rho) = \prod_{\alpha>0} (\alpha^{1/2} - \alpha^{-1/2}) \tag{5}$$

(instead of $A(\rho) = \prod_{\alpha>0}(\alpha^{1/2} - \alpha^{1/2})$).

On page 35 (§2, section 4), formula (9) should read

$$d^\circ\pi = \prod_{\alpha>0} \frac{\langle \rho + \lambda_\pi, \alpha\rangle}{\langle \rho, \alpha\rangle} \tag{9}$$

instead of having the numerator $\langle \rho + \lambda_{\pi,\alpha}\rangle$.

On page 50 (Notes, n^o 16) in the last phrase the reference should be 'See 2.5.1 in chapter IV' (instead of only 'See 2.5.1.').

References

[1] Armand Borel, *Essays in the History of Lie Groups and Algebraic Groups*, History of Mathematics, vol. 21, American Mathematical Society, Providence, 2001.