LOCAL ε-FACTORS

BERNDT E. SCHWERDTFEGER

For Petra

ABSTRACT. Note on local L-series, functional equation, ε-factors, GAUSS sums and integrals.

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1. LOCAL ζ-FUNCTIONS

1.1. Definition of L-function and Z-integral. K is a local field and $\varphi \in \mathcal{S}(K)$ a SCHWARTZ function (that is: $\mathcal{C}^\infty$ with compact support; in the non–archimedean case this basically means a finite function).

Definition 1.1.

$$Z(\lambda,\varphi) = \int_{K^\times} \lambda(t)\varphi(t)\, d_1^\times t \qquad L(\lambda) = \begin{cases} \dfrac{1}{1-\lambda(\mathfrak{p})} & \lambda \in \Lambda_K, \lambda \neq \omega_0 \\ 1 & \lambda \in \Omega_K - \Lambda_K \end{cases}$$

Theorem 1.1. *The Z integrals converge for* $\operatorname{Re}\lambda = \sigma > 0$, *in the ramified case even for all* $\lambda \in \Omega_K - \Lambda_K$. *The function* $\dfrac{Z(\lambda,\varphi)}{L(\lambda)}$ *is holomorph, which gives a meromorphic continuation of the Z-integral on all of* Ω_K.

Functional equation[1]

(1)
$$\frac{Z(\omega_1\lambda^{-1},\widehat{\varphi})}{L(\omega_1\lambda^{-1})} = \varepsilon(\lambda,\tau)\cdot\frac{Z(\lambda,\varphi)}{L(\lambda)}$$

Moreover $Z(\lambda,\varphi)$ *is holomorph for all* $\lambda \in \Omega_K$ *in case* $\varphi(0) = 0$ *(i.e. if* $\varphi \in \mathcal{S}(K^\times)$*).*

1.2. Calculation of Z-function and the ε-factor. Let $\varphi \in \mathcal{S}(K)$, then $\exists\, \ell \le k$ with $\varphi(x+y) = \varphi(x) \quad \forall y \in \mathfrak{p}^k$ and support $\operatorname{supp}\varphi \subset \mathfrak{p}^\ell$.

Let φ_k be the characteristic function of $\mathfrak{p}^k$, $\varphi = \sum \varphi(x) T_x\varphi_k$, where $(T_x\varphi)(y) = \varphi(y-x)$ and $x \in \mathfrak{p}^\ell$ runs through representatives modulo $\mathfrak{p}^k$. Without restriction $\varphi = T_x\varphi_k$ with $\operatorname{ord} x \le k$.

Lemma 1.2.
$$\widehat{T_x\varphi_k} = \tau_x \cdot \varphi_{-k-\delta}\cdot q^{-k-\delta/2}$$
with $\delta = \operatorname{ord}\tau$ *as in* WEIL [1, def. 4 in II, §5].

Proof. By calculating
$$\widehat{T_x\varphi_k}(y) = \int_K \varphi_k(z-x)\tau(yz)\,dz = \tau(yx)\int_K \varphi_k(z)\tau(yz)\,dz = (\tau_x\cdot\widehat{\varphi}_k)(y)$$

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[1] for τ see section 2.1

and

$$(\tau_x \cdot \widehat{\varphi}_k)(y) = \int_{\mathfrak{p}^k} \tau(yx)\,dx = \begin{cases} 0 & \tau_y|\mathfrak{p}^k \neq 1 \\ \mathrm{vol}(\mathfrak{p}^k) & \tau_y|\mathfrak{p}^k = 1 \end{cases}$$

$$\tau_y|\mathfrak{p}^k = 1 \Longleftrightarrow y \in \mathfrak{p}^{-\delta-k}$$

$$\mathrm{vol}_\tau(\mathfrak{p}^k) = q^{-k} \cdot \mathrm{vol}_\tau(\mathfrak{o}) = q^{-k-\delta/2}$$

concluding the proof. □

We are going to calculate the Z-integrals $Z(\lambda, T_x\varphi_k)$, $Z(\lambda, \tau_x\varphi_k)$ and the assertions of theorem 1.1 will arise out of them.

Let now $\mathrm{Re}\,\lambda = \sigma > 0$.

$$Z(\lambda, \varphi_k) = \int_{0\neq|t|\leq q^{-k}} \lambda(t)\,d_1^\times t = \Big(\sum_{n\geq k} \lambda(\pi^n)\Big) \cdot \int_{\mathfrak{o}^\times} \lambda(t)\,d_1^\times t =$$

$$= \frac{\lambda(\mathfrak{p})^k}{1-\lambda(\mathfrak{p})} \quad \text{for } \lambda \in \Lambda_K,$$

$$= 0 \quad \text{otherwise } \lambda \notin \Lambda_K.$$

$$\frac{Z(\lambda,\varphi_k)}{L(\lambda)} = \lambda(\mathfrak{p})^k \quad \lambda \in \Lambda_K \tag{2}$$

Let now $x \in K^\times$ have $\mathrm{ord}\, x = \ell < k$.

$$Z(\lambda, T_x\varphi_k) = \int_{K^\times} \lambda(t)\varphi_k(t-x)\,d_1^\times t = \lambda(x) \int_{K^\times} \lambda(t)\varphi_k(x(t-1))\,d_1^\times t =$$

$$= \lambda(x) \int_{1+\mathfrak{p}^{k-\ell}} \lambda(t)\,d_1^\times t$$

$$\text{as } \varphi_k(x(t-1)) \neq 0 \Longleftrightarrow x(t-1) \in \mathfrak{p}^k \Longleftrightarrow t-1 \in \mathfrak{p}^{k-\ell} \Longleftrightarrow t \in 1+\mathfrak{p}^{k-\ell}$$

Now $\lambda|1+\mathfrak{p}^{k-\ell} = 1 \Longleftrightarrow f(\lambda) \leq k-\ell$ so that we get

$$Z(\lambda, T_x\varphi_k) = \begin{cases} \lambda(x) \cdot q^{1+\ell-k} \cdot (q-1)^{-1} & \text{for } f(\lambda) \leq k-\ell \\ 0 & \text{otherwise } (f(\lambda) > k-\ell). \end{cases} \tag{3}$$

Under the same assumption

$$Z(\lambda, \tau_x\varphi_{-k-\delta}) = \int_{K^\times} \lambda(t)\tau(xt)\varphi_{-k-\delta}(t)\,d_1^\times t =$$

$$= \sum_{n\geq -k-\delta} \lambda(\pi^n) \int_{\mathfrak{o}^\times} \lambda(t)\tau(\pi^n xt)\,d_1^\times t = \sum_{n\geq -k-\delta} \lambda(\pi)^n \cdot \gamma(\lambda, \pi^n x)$$

From prop. 2.1 we collect the results for $\lambda \in \Lambda_K$

$$\ell + n \geq -\delta \qquad \gamma(\lambda, \pi^n x) = 1$$

$$\ell + n = -\delta - 1 \qquad \gamma(\lambda, \pi^n x) = \frac{1}{1-q}$$

$$\ell + n < -\delta - 1 \qquad \gamma(\lambda, \pi^n x) = 0$$

For $\lambda \in \Lambda_K$

$$Z(\lambda, \tau_x\varphi_{-k-\delta}) = \lambda(\mathfrak{p})^{-1-\ell-\delta} \frac{1}{1-q} + \frac{\lambda(\mathfrak{p})^{-\ell-\delta}}{1-\lambda(\mathfrak{p})} =$$

$$= \lambda(\mathfrak{p})^{-1-\ell-\delta} L(\lambda) \frac{1-q\lambda(\mathfrak{p})}{1-q}$$

$$Z(\lambda, \tau_x\varphi_{-k-\delta}) = \lambda(\mathfrak{p})^{-1-\ell-\delta} \cdot \frac{L(\lambda) \cdot L(\omega_{-1})}{L(\omega_{-1}\lambda)} \tag{4}$$

For $\lambda \notin \Lambda_K$, $\gamma(\lambda, x\pi^n) \neq 0 \iff n+\ell = -\delta - f(\lambda)$. For $f(\lambda) > k-\ell$ is $n < -\delta - k$, this does not occur and $Z(\lambda, \tau_x\varphi_{-k-\delta}) = 0$. So, let $0 \neq f(\lambda) \leq k-\ell$, we have exactly one summand corresponding to $n = -\ell - f(\lambda) - \delta$:

$$Z(\lambda, \tau_x\varphi_{-k-\delta}) = \lambda(\pi)^{-\ell-f(\lambda)-\delta} \cdot \gamma(\lambda, x\pi^{-\delta-f(\lambda)-\ell})$$

and feeding the results in case $f(\lambda) > 0$ from prop. 2.1

$$\begin{aligned} Z(\lambda, \tau_x\varphi_{-k-\delta}) &= \lambda(\pi)^{-\ell-f(\lambda)-\delta} \cdot \frac{q^{-f(\lambda)/2}}{1-q^{-1}} \cdot \bar{\kappa} \cdot \overline{\lambda(-x\pi^{-\ell})} = \\ &= \lambda(\pi)^{-\delta-f(\lambda)}\lambda(-1)\lambda(x)^{-1}\bar{\kappa} \cdot \frac{q^{-f(\lambda)/2}}{1-q^{-1}} \end{aligned} \tag{5}$$

where $\bar{\kappa} = \lambda(-1)q^{-f(\lambda)/2}G(\lambda, \pi^{\delta+f(\lambda)})$ is a normalized GAUSS sum, see section 2.

We are now ready to *verify the functional equation* (1) in the two cases $\varphi = \varphi_k$ and $\varphi = T_x\varphi_k$ with $\operatorname{ord} x = \ell < k$.

First the case $\varphi = \varphi_k$. For $\lambda \notin \Lambda_K$ both sides of (1) are zero. For $\lambda \in \Lambda_K$ we get by (2)

$$\frac{Z(\omega_1\lambda^{-1}, \widehat{\varphi_k})}{L(\omega_1\lambda^{-1})} = q^{-k-\delta/2} \cdot \omega_1(\mathfrak{p})^{-k-\delta}\lambda(\mathfrak{p})^{k+\delta} = q^{\delta/2} \cdot \lambda(\mathfrak{p})^{\delta} \cdot \frac{Z(\lambda, \varphi_k)}{L(\lambda)}$$

verifying (1) with the local ε-factor

$$\varepsilon(\lambda, \tau) = (\omega_{-1/2}\lambda)(\mathfrak{p})^{\delta}, \qquad \lambda \in \Lambda_K \tag{6}$$

Next the case $\varphi = T_x\varphi_k$ with $\operatorname{ord} x = \ell < k$. The functional equation is again valid for $f(\lambda) > k-\ell$, as both sides vanish. For $\lambda \in \Lambda_K$, by Lemma 1.2 and (4)

$$\begin{aligned} \frac{Z(\omega_1\lambda^{-1}, \widehat{\varphi})}{L(\omega_1\lambda^{-1})} &= q^{-k-\delta/2} \cdot (\omega_1\lambda^{-1})(\mathfrak{p})^{-1-\ell-\delta} \cdot \frac{L(\omega_{-1})}{L(\lambda^{-1})} = \\ &= q^{-k-\delta/2} \cdot q^{1+\ell+\delta} \cdot \lambda(\mathfrak{p})^{1+\ell+\delta} \cdot \frac{1-\lambda^{-1}(\mathfrak{p})}{1-q} = \end{aligned}$$

as $\lambda(x) = \lambda(\mathfrak{p})^{\ell}$

$$= q^{\delta/2} \cdot \lambda(\mathfrak{p})^{\delta} \cdot \lambda(x) \cdot q^{1+\ell-k} \cdot \frac{1-\lambda(\mathfrak{p})}{q-1} =$$

by (6) and (3)

$$= \varepsilon(\lambda, \tau) \cdot \frac{Z(\lambda, \varphi)}{L(\lambda)}$$

Now let λ be ramified with $f(\lambda) \leq k-\ell$, by Lemma 1.2 and (5)

$$\begin{aligned} Z(\omega_1\lambda^{-1}, \widehat{\varphi}) &= q^{-k-\delta/2} Z(\omega_1\lambda^{-1}, \tau_x\varphi_{-k-\delta}) = \\ &= q^{-k-\delta/2} q^{\delta+f(\lambda)} \lambda(\pi)^{\delta+f(\lambda)} \lambda(-1)|x|^{-1}\lambda(x)\bar{\kappa}(\lambda^{-1}) \cdot \frac{q^{-f(\lambda)/2}}{1-q^{-1}} = \\ &= q^{\delta/2+f(\lambda)/2} \cdot \lambda(\pi)^{\delta+f(\lambda)}\lambda(-1)\bar{\kappa}(\lambda^{-1}) \cdot q^{1+\ell-k}\lambda(x) \cdot (q-1)^{-1} \end{aligned}$$

as $\lambda(-1)\bar{\kappa}(\lambda^{-1}) = \kappa(\lambda)$ and using (3)

$$= (\omega_{-1/2}\lambda)(\pi)^{\delta+f(\lambda)} \cdot \kappa(\lambda) \cdot Z(\lambda, \varphi)$$

which proves (1) with the local ε-factor in the ramified case

$$\varepsilon(\lambda, \tau) = (\omega_{-1/2}\lambda)(\pi)^{\delta} \cdot \kappa(\lambda), \qquad \lambda \notin \Lambda_K \tag{7}$$

2. GAUSS INTEGRALS AND GAUSS SUMS

2.1. **Definition.** Let $\tau : K \longrightarrow \mathbf{S}^1$ be a fixed character and $d\tau$ the self dual HAAR measure on K relative to τ. Let $\delta = \operatorname{ord}\tau$ be the order and $d \in K^\times$ such that $\operatorname{ord} d = \delta$, see WEIL [1, def. 4 in II, §5].

Let $\omega \in \Omega(K)$ be of *conductor* $\mathfrak{p}^f$, see WEIL [1, def. 7 in VII, §3], write its exponent $f = f(\omega)$, let $c \in K^\times$ be such that $\operatorname{ord} c = f$, set $b = c \cdot d$.

The GAUSS *sum* is defined by

Definition 2.1.

$$G(\omega, b) = \sum_{\varepsilon \in \mathfrak{o}^\times / 1 + \mathfrak{p}^f} \omega(\varepsilon)\tau(b^{-1}\varepsilon)$$

This is well defined as for $y \in \mathfrak{p}^f = c \cdot \mathfrak{o}$ we have $b^{-1} y \in d^{-1}\mathfrak{o}$, hence $\tau(b^{-1}(\varepsilon + y)) = \tau(b^{-1}\varepsilon)$ and $\omega(\varepsilon + y) = \omega(\varepsilon) \cdot \omega(1 + \varepsilon^{-1} y) = \omega(\varepsilon)$

The GAUSS *integral* is defined by

Definition 2.2.

$$\gamma(\omega, x) = \int_{\mathfrak{o}^\times} \omega(t)\tau(tx)\, d_1^\times t$$

where $d_1^\times t$ is normed to $\operatorname{vol}(\mathfrak{o}^\times) = 1$.

We define $\varphi_\omega : K \longrightarrow \mathbf{S}^1$ by $\varphi_\omega | \mathfrak{o}^\times = \omega | \mathfrak{o}^\times$ and $= 0$ elsewhere.

Remark. Formally $\gamma(\omega, x)$ is the mutliplicative FOURIER coefficient of τ_x.

Proposition 2.1. *We have*

$$f = 0 \qquad \gamma(\omega, x) = \begin{cases} 1 & \operatorname{ord} x \ge -\delta \\ \dfrac{-1}{q-1} & \operatorname{ord} x = -\delta - 1 \\ 0 & \operatorname{ord} x < -\delta - 1 \end{cases}$$

$$f > 0 \qquad \gamma(\omega, x) = \frac{|c|^{1/2}}{1 - q^{-1}} \cdot \kappa^{-1} \cdot \overline{\varphi_\omega(-bx)}$$

with $\kappa = |b|^{-1/2} \cdot \widehat{\varphi}_{\bar{\omega}}(b^{-1}) = \omega(-1)|c|^{1/2}\overline{G(\omega, b)}$ *a* normalized GAUSS *sum,* $\kappa \in \mathbf{S}^1$, *see* WEIL [1, VII, §7, prop. 13].

2.2. **Calculation.** The proof proceeds by playing back the calculation to the additive FOURIER transform and utilising self duality. Of course the unramified case is clear. Thus let $f \ge 1$. As is well-known $d_1^\times t = \dfrac{|d|^{-1/2}}{1 - q^{-1}} \cdot \dfrac{d\tau(t)}{|t|}$. Hence

$$\gamma(\omega, x) = \frac{|d|^{-1/2}}{1 - q^{-1}} \cdot \widehat{\varphi}_\omega(x)$$

As the support of φ_ω is $\mathfrak{o}^\times \subset \mathfrak{o}$ and φ_ω is constant on classes modulo $\mathfrak{p}^f = c \cdot \mathfrak{o}$, the support of $\widehat{\varphi}_\omega$ is in $\mathfrak{p}^{-\delta - f} = b^{-1} \cdot \mathfrak{o}$ and constant modulo $d^{-1} \cdot \mathfrak{o}$.

Assertion 2.1.

$$\operatorname{supp} \widehat{\varphi}_\omega = b^{-1}\mathfrak{o}^\times$$

Proof. Let $x \in b^{-1}\mathfrak{p}$, we will show $\widehat{\varphi}_\omega(x) = 0$. If $f = 1$, $\widehat{\varphi}_\omega(x) = \int_{\mathfrak{o}^\times} \omega(t)\, d\tau(t) = 0$, as ω is non-trivial.

For $f > 1$ let $R \subset \mathfrak{o}^\times$ be a system of representatives $\mathfrak{o}^\times = R(1 + \mathfrak{p}^{f-1})$:

$$\widehat{\varphi}_\omega(x) = \sum_{\rho \in R} \int_{\mathfrak{p}^{f-1}} \varphi_\omega(\rho + t)\tau((\rho + t)x)\, d\tau(t) = \sum_{\rho \in R} \omega(\rho)\tau(\rho x) \int_{\mathfrak{p}^{f-1}} \omega(1 + t)\, d\tau(t)$$

which again = 0 as the last integral vanishes. □

Now $\widehat{\varphi}_\omega(x\varepsilon) = \omega(\varepsilon^{-1})\widehat{\varphi}_\omega(x)$ for $\varepsilon \in \mathfrak{o}^\times$, $\Longrightarrow \quad \widehat{\varphi}_\omega(x) = \widehat{\varphi}_\omega(b^{-1})\overline{\varphi_\omega(bx)} \quad \forall\, x \in K$.
By development we get $\widehat{\varphi}_\omega(b^{-1}) = \sum_{\mathfrak{o}^\times \bmod \mathfrak{p}^f} \omega(\varepsilon)\tau(\varepsilon b^{-1})\cdot\mathrm{vol}_{d\tau}(\mathfrak{p}^f)$, therefore is $\widehat{\varphi}_\omega(b^{-1}) = G(\omega, b)\cdot|c|\cdot|d|^{1/2}$.

By FOURIER inversion follows

$$\begin{aligned}
\varphi_\omega(t) &= \int_K \widehat{\varphi}_\omega(x)\tau(-tx)d\tau(x) = \widehat{\varphi}_\omega(b^{-1})\int_K \overline{\varphi_\omega(bx)\tau(tx)}d\tau(x) = \\
&= \widehat{\varphi}_\omega(b^{-1})\cdot|b|^{-1}\int_K \overline{\varphi_\omega(x)\tau(b^{-1}tx)}d\tau(x) = \\
&= \widehat{\varphi}_\omega(b^{-1})\cdot|b|^{-1}\overline{\widehat{\varphi}_\omega(b^{-1}t)} = \\
&= |b|^{-1}\cdot\widehat{\varphi}_\omega(b^{-1})\cdot\overline{\widehat{\varphi}_\omega(b^{-1})}\cdot\varphi_\omega(t)
\end{aligned}$$

which already implies $\kappa\cdot\bar{\kappa} = 1$. As $\widehat{\varphi}_{\bar{\omega}}(x) = \omega(-1)\overline{\widehat{\varphi}_\omega(x)}$ the explicit formulas of prop. 2.1 follow.

REFERENCES

[1] André Weil, *Basic Number Theory*, Classics in Mathematics, Springer, Berlin, Heidelberg, New York, 1973, 1995.