NOTE ON ADJOINT FUNCTORS

BERNDT E. SCHWERDTFEGER

Abstract. This note summarizes the basics on adjoint functors.

Preface

This note gives the basic definitions of *categories*, *functors* between categories and *natural transformations* of functors (*morphisms of functors*). In particular it describes the relation of *adjunction* between two functors $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$.

Berlin, April 24, 2011

1. Categories and functors

1.1. Categories.

Definition 1.1. A *category* $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, \circ)$ is given by

- a set[1] of *objects* $\mathcal{C}_0$
- a set of *arrows* $\mathcal{C}_1$ (or *morphisms*)
- two maps $s, t : \mathcal{C}_1 \rightrightarrows \mathcal{C}_0$, called *source* and *target*.

An arrow $f \in \mathcal{C}_1$ such that $s(f) = A$ and $t(f) = B$ is usually noted $A \xrightarrow{f} B$ or $f : A \to B$. For two objects A, B the set of arrows between them is denoted

$$\mathrm{Mor}_{\mathcal{C}}(A, B) = \{f \in \mathcal{C}_1 \mid f : A \to B\}$$

Thus we have the disjoint union $\mathcal{C}_1 = \bigcup \mathrm{Mor}_{\mathcal{C}}(A, B)$, where (A, B) runs over all pairs of objects in $\mathcal{C}_0$. We also let $\mathrm{Ob}\,\mathcal{C} = \mathcal{C}_0$ and $\mathrm{Ar}\,\mathcal{C} = \mathcal{C}_1$.

The last item in the list of a category is a *composition* $\circ$

(1) For any three objects A, B, C there is a composition map

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(A, B) \times \mathrm{Mor}_{\mathcal{C}}(B, C) &\to \mathrm{Mor}_{\mathcal{C}}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

satisfying $(f \circ g) \circ h = f \circ (g \circ h)$ whenever that makes sense.

(2) For each object A there exists an element $id_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ that satisfies $id_B \circ f = f \circ id_A = f$ for $f : A \to B$.

Date: 2011-04-25.

2010 *Mathematics Subject Classification.* Primary 18A40; Secondary 18A30.

Key words and phrases. Categories, morphisms, functors, adjunction.

[1] in some *universe*, see [1, I.1] or [2, I, appendice]

Situations where $f : A \to B$, $g : B \to C$ and $h = g \circ f : A \to C$ are depicted in such diagrams:
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$
or
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "h"'] & B \arrow[r, "g"] & C \end{tikzcd}.$$
The arrows are usually described by the sets $\mathrm{Mor}_{\mathcal{C}}(A, B)$ and composition is clear from the context.

Note. For sets recall the *fiber product* of two maps $s : X \to Z$ and $t : Y \to Z$: $X \underset{Z}{\times} Y = \{(x, y) \in X \times Y \mid s(x) = t(y)\}$. The composition map in a category can thus also be described as a map

$$\begin{aligned} \circ : \mathcal{C}_1 \underset{\mathcal{C}_0}{\times} \mathcal{C}_1 &\longrightarrow \mathcal{C}_1 \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

together with the associativity condition above. The fiber product is defined by the source and target maps s, t in the definition of a category.

Example 1.1. Let $\mathcal{U}$ be a universe and take $\mathrm{Ob}\,\mathcal{C} = \mathcal{U}$. For any sets in the universe $x, y \in \mathcal{U}$ the set of mappings $\mathrm{Map}(x, y)$ will be taken as morphisms: $\mathrm{Mor}_{\mathcal{C}}(x, y) = \mathrm{Map}(x, y)$, which is a member of the universe $\mathcal{U}$, with the obvious composition. This *category of sets* will be denoted $\mathcal{S}ets$. This category is not itself a member of the universe $\mathcal{U}$, but of $\mathcal{V}$, if $\mathcal{U}$ is a member of the universe $\mathcal{V}$, $\mathcal{U} \in \mathcal{V}$. We usually drop the reference to the universe.

Example 1.2. The *dual* category $\mathcal{A}^\circ$ is defined by $\mathrm{Ob}\,\mathcal{A}^\circ = \mathrm{Ob}\,\mathcal{A}$ and for two objects $A, B \in \mathrm{Ob}\,\mathcal{A}$ the morphisms are $\mathrm{Mor}_{\mathcal{A}^\circ}(A, B) = \mathrm{Mor}_{\mathcal{A}}(B, A)$, i.e. arrows are reversed.

Example 1.3. The *category of groups* is given by letting $\mathrm{Ob}\,\mathcal{G}rp$ be the set of groups and $\mathrm{Mor}_{\mathcal{G}rp}(G, H) = \{f : G \to H \mid f(x \cdot y) = f(x) \cdot f(y)\ \forall\ x, y \in G\}$, i.e. they are the usual *group homomorphisms*. The groups A such that $x \cdot y = y \cdot x$ form the *sub–category* of abelian groups $\mathcal{A}b$ in an obvious sense. For abelian groups A, B we have $\mathrm{Mor}_{\mathcal{A}b}(A, B) = \mathrm{Mor}_{\mathcal{G}rp}(A, B)$.

More examples for any type of algebraic structure $\mathcal{A}$ can be constructed. In this context the morphisms are called *homomorphisms* and their set is noted $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ instead of $\mathrm{Mor}_{\mathcal{A}}(X, Y)$.

A morphism $f : A \to B$ is an *isomorphism* if there is a morphism $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$. A and B are called *isomorphic*, denoted $A \simeq B$ or $A \xrightarrow{\sim} B$.

For $A = B$ a morphism is called *endomorphism* and $\mathrm{End}_{\mathcal{C}}(A) := \mathrm{Mor}_{\mathcal{C}}(A, A)$. An isomorphism in that set is called *automorphism* and $\mathrm{Aut}_{\mathcal{C}}(A) \subset \mathrm{End}_{\mathcal{C}}(A)$ is a group with id_A as neutral element.

A morphism $f : A \to B$ is a *monomorphism* (resp. *epimorphism*) provided that for all objects X (resp. Y) the map

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(X, A) &\to \mathrm{Mor}_{\mathcal{C}}(X, B) \\ x &\mapsto f \circ x \end{aligned} \qquad \text{resp.} \qquad \begin{aligned} \mathrm{Mor}_{\mathcal{C}}(B, Y) &\to \mathrm{Mor}_{\mathcal{C}}(A, Y) \\ y &\mapsto y \circ f \end{aligned}$$

$$\begin{tikzcd} X \arrow[d, "x"'] \arrow[dr, "fx"] & \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "yf"'] & B \arrow[d, "y"] \\ & Y \end{tikzcd}$$

is *injective*.

This is sometimes depicted $A \overset{f}{\hookrightarrow} B$ resp. $A \overset{f}{\twoheadrightarrow} B$.

Exercise 1.1. Verify that for the category $\mathcal{S}ets$ this categorical meaning coincides with the usual meaning: monomorphisms are injective and epimorphisms are surjective.

An object F is called *final* if $\operatorname{card} \operatorname{Mor}_{\mathcal{C}}(X, F) = 1$ for any X. An object I is called *inital* if $\operatorname{card} \operatorname{Mor}_{\mathcal{C}}(I, Y) = 1$ for any Y.

Obviously all initial objects are isomorphic, as are all final objects. There need not exist any, though.

If $S \in \operatorname{Ob} \mathcal{C}$ is any object of $\mathcal{C}$ the category $\mathcal{C}/S$ of *objects above* S is given by the morphisms with target S: $\operatorname{Ob} \mathcal{C}/S = \{f : X \to S\}$ with morphisms between $f : X \to S$ and $g : Y \to S$ given by $\operatorname{Mor}_{\mathcal{C}/S}(f, g) = \{h : X \to Y \mid f = g \circ h\}$, or as diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

1.2. **Functors.**

Definition 1.2. A *covariant functor* $u : \mathcal{A} \to \mathcal{B}$ is a map on the underlying structural sets, i.e. maps $u_i : \mathcal{A}_i \to \mathcal{B}_i$ $(i = 0, 1)$, commuting with *source*, *target* and *composition* maps: $s \circ u_1 = u_0 \circ s$, $t \circ u_1 = u_0 \circ t$, $u_1(g \circ f) = u_1(g) \circ u_1(f)$ and $u_1(id_A) = id_{u_0(A)}$.

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "u_1"] \arrow[d, "s"'] & \mathcal{B}_1 \arrow[d, "s"] \\ \mathcal{A}_0 \arrow[r, "u_0"'] & \mathcal{B}_0 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_1 \arrow[r, "u_1"] \arrow[d, "t"'] & \mathcal{B}_1 \arrow[d, "t"] \\ \mathcal{A}_0 \arrow[r, "u_0"'] & \mathcal{B}_0 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_1 \underset{\mathcal{A}_0}{\times} \mathcal{A}_1 \arrow[r, "u_1 \times u_1"] \arrow[d, "\circ"'] & \mathcal{B}_1 \underset{\mathcal{B}_0}{\times} \mathcal{B}_1 \arrow[d, "\circ"] \\ \mathcal{A}_1 \arrow[r, "u_1"'] & \mathcal{B}_1 \end{tikzcd}$$

The map on objects and the map on arrows of a functor is, by abuse of notation, usually denoted by the same symbol, i.e. a functor $u : \mathcal{A} \to \mathcal{B}$ is described by a map $u : \mathcal{A}_0 \to \mathcal{B}_0$ and, for each $X, Y \in \mathcal{A}_0$, by $u : \operatorname{Mor}_{\mathcal{A}}(X, Y) \to \operatorname{Mor}_{\mathcal{B}}(u(X), u(Y))$.

If in the above definition we replace $\mathcal{A}$ by the dual $\mathcal{A}^\circ$ the functor is called *contravariant.* It is thus characterized by the fact that a morphism $f : X \to Y$ is transformed into $u(f) : u(Y) \to u(X)$, the direction of the arrows being reversed.

Example 1.4. In a category $\mathcal{C}$ the sets of morphisms $\operatorname{Mor}_{\mathcal{C}}(X, Y)$ define a natural functor, from $\mathcal{C}^\circ \times \mathcal{C}$ to $\mathcal{S}ets$, which is *contravariant* in X and *covariant* in Y, in the following way: for $f : X' \to X$ and $g : Y \to Y'$ define

$$\begin{aligned} \operatorname{Mor}(f, Y) : \operatorname{Mor}_{\mathcal{C}}(X, Y) &\to \operatorname{Mor}_{\mathcal{C}}(X', Y) \\ h &\mapsto h \circ f \end{aligned}$$

and

$$\begin{aligned} \operatorname{Mor}(X, g) : \operatorname{Mor}_{\mathcal{C}}(X, Y) &\to \operatorname{Mor}_{\mathcal{C}}(X, Y') \\ h &\mapsto g \circ h \end{aligned}$$

Then $\mathrm{Mor}(f,g) = \mathrm{Mor}(X',g) \circ \mathrm{Mor}(f,Y) = \mathrm{Mor}(f,Y') \circ \mathrm{Mor}(X,g)$ is the diagonal in the following (commutative) diagram:

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{C}}(X,Y) \arrow[r, "\mathrm{Mor}(f,Y)"] \arrow[d, "\mathrm{Mor}(X,g)"'] & \mathrm{Mor}_{\mathcal{C}}(X',Y) \arrow[d, "\mathrm{Mor}(X',g)"] \\ \mathrm{Mor}_{\mathcal{C}}(X,Y') \arrow[r, "\mathrm{Mor}(f,Y')"'] & \mathrm{Mor}_{\mathcal{C}}(X',Y') \end{tikzcd}$$

and the rules for a functor are readily verified. In particular: $\mathrm{Mor}(id_X, g) = \mathrm{Mor}(X,g)$ and $\mathrm{Mor}(f, id_Y) = \mathrm{Mor}(f,Y)$.

Example 1.5. In the above example we can fix X or Y and get functors

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(X,\cdot) : \mathcal{C} &\longrightarrow \mathcal{S}ets \\ Y &\longmapsto \mathrm{Mor}_{\mathcal{C}}(X,Y) \\ g &\longmapsto \mathrm{Mor}_{\mathcal{C}}(X,g) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(\cdot,Y) : \mathcal{C}^{\circ} &\longrightarrow \mathcal{S}ets \\ X &\longmapsto \mathrm{Mor}_{\mathcal{C}}(X,Y) \\ f &\longmapsto \mathrm{Mor}_{\mathcal{C}}(f,Y) \end{aligned}$$

Actually the functors vary with X resp. Y in a *functorial* way, which leads us to the definition of *morphisms of functors*.

1.3. **Morphisms of functors.**

Definition 1.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories, and $u, v : \mathcal{A} \to \mathcal{B}$ be two functors between them.

A *morphism of functors* from u to v: $\theta : u \to v$, is a family

$$(\theta_A)_A \in \prod_{A \in \mathrm{Ob}\,\mathcal{A}} \mathrm{Mor}_{\mathcal{B}}(u(A), v(A))$$

satisfying the rule that for all $A, A' \in \mathcal{A}$ and all $f : A \to A'$ we have $\theta_{A'} \circ u(f) = v(f) \circ \theta_A$

$$\begin{tikzcd} u(A) \arrow[r, "u(f)"] \arrow[d, "\theta_A"'] & u(A') \arrow[d, "\theta_{A'}"] \\ v(A) \arrow[r, "v(f)"] & v(A') \end{tikzcd}$$

The set of morphisms from u to v is denoted $\mathrm{Mor}(u,v)$.

Example 1.6. The functors in example 1.5 define morphisms of functors: let $X^\bullet = \mathrm{Mor}_{\mathcal{C}}(X,\cdot) : \mathcal{C} \to \mathcal{S}ets$ and for $f : X' \to X$ let $f^\bullet : X^\bullet \to X'^\bullet$ be the family

$$f^\bullet(Y) = \mathrm{Mor}_{\mathcal{C}}(f,Y) : \mathrm{Mor}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X',Y)$$

The rule to verify is for all Y, Y' and $g : Y \to Y'$ that

$$f^\bullet(Y') \circ X^\bullet(g) = X'^\bullet(g) \circ f^\bullet(Y)$$

which is exactly the relation verified in example 1.4: as $f^\bullet(Y) = \mathrm{Mor}_{\mathcal{C}}(f,Y)$ and $X^\bullet(g) = \mathrm{Mor}_{\mathcal{C}}(X,g)$

Example 1.7. Similarly, the second example in example 1.5 defines morphisms of functors $g_\bullet : Y_\bullet \to Y'_\bullet$ for $g : Y \to Y'$, where $Y_\bullet = \mathrm{Mor}_{\mathcal{C}}(\cdot, Y) : \mathcal{C}^\circ \to \mathcal{S}ets$

$$g_\bullet(X) = \mathrm{Mor}(X, g) : \mathrm{Mor}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, Y')$$

The verification of functoriality is left to the reader.

Note. In the above examples $g_\bullet(X) = X^\bullet(g)$ and $f^\bullet(Y) = Y_\bullet(f)$. For $h : X \to Y$ we have $g_\bullet(X)h = g \circ h$ and $f^\bullet(Y)h = h \circ f$.

Lemma 1.1. (Yoneda)
Let $\mathcal{C}$ be a category, $Y \in \mathrm{Ob}\,\mathcal{C}$ an object in $\mathcal{C}$ and $u : \mathcal{C}^\circ \to \mathcal{S}ets$ be a contravariant functor. We have

$$u(Y) \simeq \mathrm{Mor}(Y_\bullet, u)$$

Proof. We will construct canonical mappings in both directions. Let $\theta \in \mathrm{Mor}(Y_\bullet, u)$, then $\theta(Y) : Y_\bullet(Y) \to u(Y)$ and we let $y = \theta(Y)id_Y \in u(Y)$ correspond to θ.

In the other direction start with $y \in u(Y)$. We need to define, for any X, a family of maps $\theta(X) : Y_\bullet(X) \to u(X)$ and we simply take $\theta(X)f = u(f)y$. As u is a (contravariant) functor, we have, for $f \in Y_\bullet(X) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$, morphisms $u(f) : u(Y) \to u(X)$.

It is straight forward to verify that $\theta \leftrightarrow y$ correspond to each other. □

By duality ($\mathcal{C}$ replaced by $\mathcal{C}^\circ$) we have as well the dual

Lemma 1.2. *Let $\mathcal{C}$ be a category, $X \in \mathrm{Ob}\,\mathcal{C}$ an object in $\mathcal{C}$ and $u : \mathcal{C} \to \mathcal{S}ets$ be a covariant functor. We have*

$$u(X) \simeq \mathrm{Mor}(X^\bullet, u)$$

Proof. Let us only note the correspondence $x \in u(X)$ to $\theta \in \mathrm{Mor}(X^\bullet, u)$: $x = u(X)id_X$ and $\theta(Y)f = u(f)x$ for $f \in X^\bullet(Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$. □

1.4. Category of functors.

Definition 1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. The *category of functors* $\mathcal{F}un(\mathcal{A}, \mathcal{B})$ is defined by the *objects* $\mathrm{Ob}\,\mathcal{F}un(\mathcal{A}, \mathcal{B}) = \{u : \mathcal{A} \to \mathcal{B} \mid$ are the *functors*$\}$ and the *morphisms* $\mathrm{Mor}_{\mathcal{F}un(\mathcal{A},\mathcal{B})}(u, v) = \{\theta : u \to v \mid$ are the *morphisms of functors*$\}$.

Definition 1.5. A functor $u : \mathcal{A} \to \mathcal{B}$ is called *faithful* (resp. *fully faithful*) if for all objects $X, Y \in \mathrm{Ob}\,\mathcal{A}$ the maps on the morphisms are *injective* (resp. *bijective*): $\mathrm{Mor}_{\mathcal{A}}(X, Y) \hookrightarrow \mathrm{Mor}_{\mathcal{B}}(u(X), u(Y))$ (resp. $\mathrm{Mor}_{\mathcal{A}}(X, Y) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{B}}(u(X), u(Y))$).

Theorem 1.3. *Let $\mathcal{C}$ be a category. Then we have fully faithful functors:*

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathcal{F}un(\mathcal{C}^\circ, \mathcal{S}ets) \qquad & \mathcal{C}^\circ &\longrightarrow \mathcal{F}un(\mathcal{C}, \mathcal{S}ets) \\ Y &\longmapsto Y_\bullet & X &\longmapsto X^\bullet \\ g &\longmapsto g_\bullet & f &\longmapsto f^\bullet \end{aligned}$$

Proof. We will only prove the first statement. The second follows by duality $\mathcal{C}$ replaced by $\mathcal{C}^\circ$.

We have to show, for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, the bijectivity of

$$\mathrm{Mor}_{\mathcal{C}}(Y, X) \longrightarrow \mathrm{Mor}(Y_\bullet, X_\bullet)$$

In fact, this has been shown in the Yoneda Lemma 1.1 with $u = X_\bullet$. □

Note. The category $\widehat{\mathcal{C}} = \mathcal{F}un(\mathcal{C}^\circ, \mathcal{S}ets)$ is called *category of presheaves of sets*. The theorem embeds $\mathcal{C} \subset \widehat{\mathcal{C}}$ as a full subcategory.

1.5. **Adjoint functors.** Let $\mathcal{A}, \mathcal{B}$ be categories and $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be two functors. We now write $\mathrm{Hom}_{\mathcal{A}}, \mathrm{Hom}_{\mathcal{B}}$ for the morphisms in the categories to distinguish them from the morphisms of functors Mor.

We consider the two functors $\mathrm{Hom}_{\mathcal{A}}(\cdot, v\cdot), \mathrm{Hom}_{\mathcal{B}}(u\cdot, \cdot) : \mathcal{A}^\circ \times \mathcal{B} \to \mathcal{S}ets$ and their morphisms as well as the two morphisms and their morphisms

$$u \circ v, id_{\mathcal{B}} : \mathcal{B} \to \mathcal{B} \qquad\qquad id_{\mathcal{A}}, v \circ u : \mathcal{A} \to \mathcal{A}$$

Theorem 1.4. *We have isomorphisms*

$$\begin{aligned} \mathrm{Mor}(\mathrm{Hom}_{\mathcal{A}}(\cdot, v\cdot), \mathrm{Hom}_{\mathcal{B}}(u\cdot, \cdot)) &\xrightarrow{\sim} \mathrm{Mor}(u \circ v, id_{\mathcal{B}}) \\ \varphi &\longmapsto \phi \\ \mathrm{Mor}(\mathrm{Hom}_{\mathcal{B}}(u\cdot, \cdot), \mathrm{Hom}_{\mathcal{A}}(\cdot, v\cdot)) &\xrightarrow{\sim} \mathrm{Mor}(id_{\mathcal{A}}, v \circ u) \\ \vartheta &\longmapsto \theta \end{aligned}$$

with ϕ defined by $\phi(B) = \varphi(v(B), B) id_{v(B)} : u(v(B)) \to B$ and correspondingly in the dual situation.

Proof. The proof is straightforward. If $\phi : u \circ v \to id_{\mathcal{B}}$ is given we define

$$\begin{aligned} \varphi(A, B) : \mathrm{Hom}_{\mathcal{A}}(A, v(B)) &\longrightarrow \mathrm{Hom}_{\mathcal{B}}(u(A), B) \\ f &\longmapsto \phi(B) \circ u(f) \end{aligned}$$

The verification is trivial and left to the reader. See also [1, I.7, lemme 1, 2]. □

Keeping the notation of theorem 1.4, let morphisms $\varphi, \phi, \vartheta, \theta$ be given

$$\mathrm{Hom}_{\mathcal{A}}(\cdot, v\cdot) \xrightarrow{\varphi} \mathrm{Hom}_{\mathcal{B}}(u\cdot, \cdot) \qquad\qquad \phi : u \circ v \to id_{\mathcal{B}}$$
$$\mathrm{Hom}_{\mathcal{A}}(\cdot, v\cdot) \xleftarrow{\vartheta} \mathrm{Hom}_{\mathcal{B}}(u\cdot, \cdot) \qquad\qquad \theta : id_{\mathcal{A}} \to v \circ u$$

Proposition 1.5. *We have*

$$\begin{aligned} \vartheta \circ \varphi = id &\iff id_v : v \xrightarrow{\theta v} vuv \xrightarrow{v\phi} v \\ \varphi \circ \vartheta = id &\iff id_u : u \xrightarrow{u\theta} uvu \xrightarrow{\phi u} u \end{aligned}$$

Proof. Straightforward. See also [1, I.7, prop. 7] □

Definition 1.6. Under these conditions u is called *left adjoint* (or *co-adjoint*) to v and v is called *right adjoint* (or *adjoint*) to u.

The morphisms of functors φ resp. ϑ are called *adjunction* isomorphisms, the morphisms ϕ resp. θ are called *adjunction* morphisms.

The categories $\mathcal{A}$ and $\mathcal{B}$ are called *equivalent* if the adjunction morphisms ϕ and θ are isomorphisms; the functors u and v are establishing the equivalency.

Note. Adjoints are determined up to isomorphy. An adjoint of an additive functor is additive. Adjoints are compatible with limits (e.g. kernels, products), i.e. they are left exact. Co-adjoints are compatible with co-limits (e.g. cokernels, coproducts (=sums)), i.e. they are right exact.

References

[1] Pierre Gabriel, *Des catégories Abéliennes*, Bull. Soc. math. France **90** (1962), 323-448.
[2] Alexander Grothendieck, Michael Artin, and Jean-Louis Verdier, *SGA 4 : Théorie des Topos et Cohomologie Étales des Schémas, tome 1–3*, Lecture Notes in Math., vol. 269, 270, 305, Springer, 1972.