

# NOTE ON ADJOINT FUNCTORS

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ABSTRACT. This note summarizes the basics on adjoint functors.

## PREFACE

This note gives the basic definitions of *categories*, *functors* between categories and *natural transformations* of functors (*morphisms of functors*). In particular it describes the relation of *adjunction* between two functors  $u : \mathcal{A} \rightarrow \mathcal{B}$  and  $v : \mathcal{B} \rightarrow \mathcal{A}$ .

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## 1. CATEGORIES AND FUNCTORS

### 1.1. Categories.

**Definition 1.1.** A *category*  $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, \circ)$  is given by

- a set<sup>1</sup> of *objects*  $\mathcal{C}_0$
- a set of *arrows*  $\mathcal{C}_1$  (or *morphisms*)
- two maps  $s, t : \mathcal{C}_1 \rightrightarrows \mathcal{C}_0$ , called *source* and *target*.

An arrow  $f \in \mathcal{C}_1$  such that  $s(f) = A$  and  $t(f) = B$  is usually noted  $A \xrightarrow{f} B$  or  $f : A \rightarrow B$ . For two objects  $A, B$  the set of arrows between them is denoted

$$\text{Mor}_{\mathcal{C}}(A, B) = \{f \in \mathcal{C}_1 \mid f : A \rightarrow B\}$$

Thus we have the disjoint union  $\mathcal{C}_1 = \bigcup \text{Mor}_{\mathcal{C}}(A, B)$ , where  $(A, B)$  runs over all pairs of objects in  $\mathcal{C}_0$ . We also let  $\text{Ob } \mathcal{C} = \mathcal{C}_0$  and  $\text{Ar } \mathcal{C} = \mathcal{C}_1$ .

The last item in the list of a category is a *composition*  $\circ$

- (1) For any three objects  $A, B, C$  there is a composition map

$$\begin{aligned} \text{Mor}_{\mathcal{C}}(A, B) \times \text{Mor}_{\mathcal{C}}(B, C) &\rightarrow \text{Mor}_{\mathcal{C}}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

satisfying  $(f \circ g) \circ h = f \circ (g \circ h)$  whenever that makes sense.

- (2) For each object  $A$  there exists an element  $id_A \in \text{Mor}_{\mathcal{C}}(A, A)$  that satisfies  $id_B \circ f = f \circ id_A = f$  for  $f : A \rightarrow B$ .

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<sup>1</sup>in some *universe*, see [1, I.1] or [2, I, appendice]

Situations where  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h = g \circ f : A \rightarrow C$  are depicted in such diagrams:  $A \xrightarrow{f} B$  or  $A \xrightarrow{f} B \xrightarrow{g} C$ . The arrows are usually described

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow h & \downarrow g \\ & & C \end{array}$$

by the sets  $\text{Mor}_{\mathcal{C}}(A, B)$  and composition is clear from the context.

*Note.* For sets recall the *fiber product* of two maps  $s : X \rightarrow Z$  and  $t : Y \rightarrow Z$ :  $X \times_Z Y = \{(x, y) \in X \times Y \mid s(x) = t(y)\}$ . The composition map in a category can thus also be described as a map

$$\begin{aligned} \circ : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 &\longrightarrow \mathcal{C}_1 \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

together with the associativity condition above. The fiber product is defined by the source and target maps  $s, t$  in the definition of a category.

**Example 1.1.** Let  $\mathcal{U}$  be a universe and take  $\text{Ob } \mathcal{C} = \mathcal{U}$ . For any sets in the universe  $x, y \in \mathcal{U}$  the set of mappings  $\text{Map}(x, y)$  will be taken as morphisms:  $\text{Mor}_{\mathcal{C}}(x, y) = \text{Map}(x, y)$ , which is a member of the universe  $\mathcal{U}$ , with the obvious composition. This *category of sets* will be denoted  $\mathcal{S}ets$ . This category is not itself a member of the universe  $\mathcal{U}$ , but of  $\mathcal{V}$ , if  $\mathcal{U}$  is a member of the universe  $\mathcal{V}$ ,  $\mathcal{U} \in \mathcal{V}$ . We usually drop the reference to the universe.

**Example 1.2.** The *dual category*  $\mathcal{A}^\circ$  is defined by  $\text{Ob } \mathcal{A}^\circ = \text{Ob } \mathcal{A}$  and for two objects  $A, B \in \text{Ob } \mathcal{A}$  the morphisms are  $\text{Mor}_{\mathcal{A}^\circ}(A, B) = \text{Mor}_{\mathcal{A}}(B, A)$ , i.e. arrows are reversed.

**Example 1.3.** The *category of groups* is given by letting  $\text{Ob } \mathcal{G}rp$  be the set of groups and  $\text{Mor}_{\mathcal{G}rp}(G, H) = \{f : G \rightarrow H \mid f(x \cdot y) = f(x) \cdot f(y) \forall x, y \in G\}$ , i.e. they are the usual *group homomorphisms*. The groups  $A$  such that  $x \cdot y = y \cdot x$  form the *sub-category* of abelian groups  $\mathcal{A}b$  in an obvious sense. For abelian groups  $A, B$  we have  $\text{Mor}_{\mathcal{A}b}(A, B) = \text{Mor}_{\mathcal{G}rp}(A, B)$ .

More examples for any type of algebraic structure  $\mathcal{A}$  can be constructed. In this context the morphisms are called *homomorphisms* and their set is noted  $\text{Hom}_{\mathcal{A}}(X, Y)$  instead of  $\text{Mor}_{\mathcal{A}}(X, Y)$ .

A morphism  $f : A \rightarrow B$  is an *isomorphism* if there is a morphism  $g : B \rightarrow A$  such that  $g \circ f = id_A$  and  $f \circ g = id_B$ .  $A$  and  $B$  are called *isomorphic*, denoted  $A \simeq B$  or  $A \xrightarrow{\sim} B$ .

For  $A = B$  a morphism is called *endomorphism* and  $\text{End}_{\mathcal{C}}(A) := \text{Mor}_{\mathcal{C}}(A, A)$ . An isomorphism in that set is called *automorphism* and  $\text{Aut}_{\mathcal{C}}(A) \subset \text{End}_{\mathcal{C}}(A)$  is a group with  $id_A$  as neutral element.

A morphism  $f : A \rightarrow B$  is a *monomorphism* (resp. *epimorphism*) provided that for all objects  $X$  (resp.  $Y$ ) the map

$$\begin{array}{ccc} \text{Mor}_{\mathcal{C}}(X, A) \rightarrow \text{Mor}_{\mathcal{C}}(X, B) & \text{resp.} & \text{Mor}_{\mathcal{C}}(B, Y) \rightarrow \text{Mor}_{\mathcal{C}}(A, Y) \\ x \mapsto f \circ x & & y \mapsto y \circ f \end{array}$$

$$\begin{array}{ccc} X & & \\ \downarrow x & \searrow fx & \\ A & \xrightarrow{f} & B \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \searrow yf & & \downarrow y \\ & & Y \end{array}$$

is *injective*.

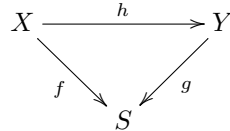
This is sometimes depicted  $A \xrightarrow{f} B$  resp.  $A \xrightarrow{f} \gg B$ .

**Exercise 1.1.** Verify that for the category *Sets* this categorical meaning coincides with the usual meaning: monomorphisms are injective and epimorphisms are surjective.

An object  $F$  is called *final* if  $\text{card Mor}_{\mathcal{C}}(X, F) = 1$  for any  $X$ . An object  $I$  is called *initial* if  $\text{card Mor}_{\mathcal{C}}(I, Y) = 1$  for any  $Y$ .

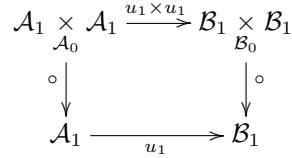
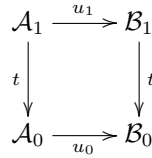
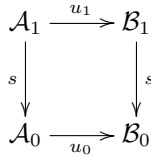
Obviously all initial objects are isomorphic, as are all final objects. There need not exist any, though.

If  $S \in \text{Ob } \mathcal{C}$  is any object of  $\mathcal{C}$  the category  $\mathcal{C}/S$  of *objects above*  $S$  is given by the morphisms with target  $S$ :  $\text{Ob } \mathcal{C}/S = \{f : X \rightarrow S\}$  with morphisms between  $f : X \rightarrow S$  and  $g : Y \rightarrow S$  given by  $\text{Mor}_{\mathcal{C}/S}(f, g) = \{h : X \rightarrow Y \mid f = g \circ h\}$ , or as diagram



### 1.2. Functors.

**Definition 1.2.** A *covariant functor*  $u : \mathcal{A} \rightarrow \mathcal{B}$  is a map on the underlying structural sets, i.e. maps  $u_i : \mathcal{A}_i \rightarrow \mathcal{B}_i$  ( $i = 0, 1$ ), commuting with *source*, *target* and *composition* maps:  $s \circ u_1 = u_0 \circ s$ ,  $t \circ u_1 = u_0 \circ t$ ,  $u_1(g \circ f) = u_1(g) \circ u_1(f)$  and  $u_1(id_A) = id_{u_0(A)}$ .



The map on objects and the map on arrows of a functor is, by abuse of notation, usually denoted by the same symbol, i.e. a functor  $u : \mathcal{A} \rightarrow \mathcal{B}$  is described by a map  $u : \mathcal{A}_0 \rightarrow \mathcal{B}_0$  and, for each  $X, Y \in \mathcal{A}_0$ , by  $u : \text{Mor}_{\mathcal{A}}(X, Y) \rightarrow \text{Mor}_{\mathcal{B}}(u(X), u(Y))$ .

If in the above definition we replace  $\mathcal{A}$  by the dual  $\mathcal{A}^\circ$  the functor is called *contravariant*. It is thus characterized by the fact that a morphism  $f : X \rightarrow Y$  is transformed into  $u(f) : u(Y) \rightarrow u(X)$ , the direction of the arrows being reversed.

**Example 1.4.** In a category  $\mathcal{C}$  the sets of morphisms  $\text{Mor}_{\mathcal{C}}(X, Y)$  define a natural functor, from  $\mathcal{C}^\circ \times \mathcal{C}$  to *Sets*, which is *contravariant* in  $X$  and *covariant* in  $Y$ , in the following way: for  $f : X' \rightarrow X$  and  $g : Y \rightarrow Y'$  define

$$\begin{aligned} \text{Mor}(f, Y) : \text{Mor}_{\mathcal{C}}(X, Y) &\rightarrow \text{Mor}_{\mathcal{C}}(X', Y) \\ h &\mapsto h \circ f \end{aligned}$$

and

$$\begin{aligned} \text{Mor}(X, g) : \text{Mor}_{\mathcal{C}}(X, Y) &\rightarrow \text{Mor}_{\mathcal{C}}(X, Y') \\ h &\mapsto g \circ h \end{aligned}$$

Then  $\text{Mor}(f, g) = \text{Mor}(X', g) \circ \text{Mor}(f, Y) = \text{Mor}(f, Y') \circ \text{Mor}(X, g)$  is the diagonal in the following (commutative) diagram:

$$\begin{array}{ccc} \text{Mor}_{\mathcal{C}}(X, Y) & \xrightarrow{\text{Mor}(f, Y)} & \text{Mor}_{\mathcal{C}}(X', Y) \\ \text{Mor}(X, g) \downarrow & & \downarrow \text{Mor}(X', g) \\ \text{Mor}_{\mathcal{C}}(X, Y') & \xrightarrow{\text{Mor}(f, Y')} & \text{Mor}_{\mathcal{C}}(X', Y') \end{array}$$

and the rules for a functor are readily verified. In particular:  $\text{Mor}(id_X, g) = \text{Mor}(X, g)$  and  $\text{Mor}(f, id_Y) = \text{Mor}(f, Y)$ .

**Example 1.5.** In the above example we can fix  $X$  or  $Y$  and get functors

$$\begin{aligned} \text{Mor}_{\mathcal{C}}(X, \cdot) : \mathcal{C} &\longrightarrow \mathcal{S}ets \\ Y &\longmapsto \text{Mor}_{\mathcal{C}}(X, Y) \\ g &\longmapsto \text{Mor}_{\mathcal{C}}(X, g) \end{aligned}$$

and

$$\begin{aligned} \text{Mor}_{\mathcal{C}}(\cdot, Y) : \mathcal{C}^{\circ} &\longrightarrow \mathcal{S}ets \\ X &\longmapsto \text{Mor}_{\mathcal{C}}(X, Y) \\ f &\longmapsto \text{Mor}_{\mathcal{C}}(f, Y) \end{aligned}$$

Actually the functors vary with  $X$  resp.  $Y$  in a *functorial* way, which leads us to the definition of *morphisms of functors*.

### 1.3. Morphisms of functors.

**Definition 1.3.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two categories, and  $u, v : \mathcal{A} \rightarrow \mathcal{B}$  be two functors between them.

A *morphism of functors* from  $u$  to  $v$ :  $\theta : u \rightarrow v$ , is a family

$$(\theta_A)_A \in \prod_{A \in \text{Ob } \mathcal{A}} \text{Mor}_{\mathcal{B}}(u(A), v(A))$$

satisfying the rule that for all  $A, A' \in \mathcal{A}$  and all  $f : A \rightarrow A'$  we have  $\theta_{A'} \circ u(f) = v(f) \circ \theta_A$

$$\begin{array}{ccc} u(A) & \xrightarrow{u(f)} & u(A') \\ \theta_A \downarrow & & \downarrow \theta_{A'} \\ v(A) & \xrightarrow{v(f)} & v(A') \end{array}$$

The set of morphisms from  $u$  to  $v$  is denoted  $\text{Mor}(u, v)$ .

**Example 1.6.** The functors in example 1.5 define morphisms of functors: let  $X^{\bullet} = \text{Mor}_{\mathcal{C}}(X, \cdot) : \mathcal{C} \rightarrow \mathcal{S}ets$  and for  $f : X' \rightarrow X$  let  $f^{\bullet} : X^{\bullet} \rightarrow X'^{\bullet}$  be the family

$$f^{\bullet}(Y) = \text{Mor}_{\mathcal{C}}(f, Y) : \text{Mor}_{\mathcal{C}}(X, Y) \longrightarrow \text{Mor}_{\mathcal{C}}(X', Y)$$

The rule to verify is for all  $Y, Y'$  and  $g : Y \rightarrow Y'$  that

$$f^{\bullet}(Y') \circ X^{\bullet}(g) = X'^{\bullet}(g) \circ f^{\bullet}(Y)$$

which is exactly the relation verified in example 1.4: as  $f^{\bullet}(Y) = \text{Mor}_{\mathcal{C}}(f, Y)$  and  $X^{\bullet}(g) = \text{Mor}_{\mathcal{C}}(X, g)$

**Example 1.7.** Similarly, the second example in example 1.5 defines morphisms of functors  $g_{\bullet} : Y_{\bullet} \rightarrow Y'_{\bullet}$  for  $g : Y \rightarrow Y'$ , where  $Y_{\bullet} = \text{Mor}_{\mathcal{C}}(\cdot, Y) : \mathcal{C}^{\circ} \rightarrow \text{Sets}$

$$g_{\bullet}(X) = \text{Mor}(X, g) : \text{Mor}_{\mathcal{C}}(X, Y) \longrightarrow \text{Mor}_{\mathcal{C}}(X, Y')$$

The verification of functoriality is left to the reader.

*Note.* In the above examples  $g_{\bullet}(X) = X^{\bullet}(g)$  and  $f^{\bullet}(Y) = Y_{\bullet}(f)$ . For  $h : X \rightarrow Y$  we have  $g_{\bullet}(X)h = g \circ h$  and  $f^{\bullet}(Y)h = h \circ f$ .

**Lemma 1.1.** (Yoneda)

Let  $\mathcal{C}$  be a category,  $Y \in \text{Ob } \mathcal{C}$  an object in  $\mathcal{C}$  and  $u : \mathcal{C}^{\circ} \rightarrow \text{Sets}$  be a contravariant functor. We have

$$u(Y) \simeq \text{Mor}(Y_{\bullet}, u)$$

*Proof.* We will construct canonical mappings in both directions. Let  $\theta \in \text{Mor}(Y_{\bullet}, u)$ , then  $\theta(Y) : Y_{\bullet}(Y) \rightarrow u(Y)$  and we let  $y = \theta(Y)id_Y \in u(Y)$  correspond to  $\theta$ .

In the other direction start with  $y \in u(Y)$ . We need to define, for any  $X$ , a family of maps  $\theta(X) : Y_{\bullet}(X) \rightarrow u(X)$  and we simply take  $\theta(X)f = u(f)y$ . As  $u$  is a (contravariant) functor, we have, for  $f \in Y_{\bullet}(X) = \text{Mor}_{\mathcal{C}}(X, Y)$ , morphisms  $u(f) : u(Y) \rightarrow u(X)$ .

It is straight forward to verify that  $\theta \leftrightarrow y$  correspond to each other.  $\square$

By duality ( $\mathcal{C}$  replaced by  $\mathcal{C}^{\circ}$ ) we have as well the dual

**Lemma 1.2.** Let  $\mathcal{C}$  be a category,  $X \in \text{Ob } \mathcal{C}$  an object in  $\mathcal{C}$  and  $u : \mathcal{C} \rightarrow \text{Sets}$  be a covariant functor. We have

$$u(X) \simeq \text{Mor}(X^{\bullet}, u)$$

*Proof.* Let us only note the correspondence  $x \in u(X)$  to  $\theta \in \text{Mor}(X^{\bullet}, u)$ :  $x = u(X)id_X$  and  $\theta(Y)f = u(f)x$  for  $f \in X^{\bullet}(Y) = \text{Mor}_{\mathcal{C}}(X, Y)$ .  $\square$

#### 1.4. Category of functors.

**Definition 1.4.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories. The *category of functors*  $\mathcal{F}un(\mathcal{A}, \mathcal{B})$  is defined by the *objects*  $\text{Ob } \mathcal{F}un(\mathcal{A}, \mathcal{B}) = \{u : \mathcal{A} \rightarrow \mathcal{B} \mid \text{are the functors}\}$  and the *morphisms*  $\text{Mor}_{\mathcal{F}un(\mathcal{A}, \mathcal{B})}(u, v) = \{\theta : u \rightarrow v \mid \text{are the morphisms of functors}\}$ .

**Definition 1.5.** A functor  $u : \mathcal{A} \rightarrow \mathcal{B}$  is called *faithful* (resp. *fully faithful*) if for all objects  $X, Y \in \text{Ob } \mathcal{A}$  the maps on the morphisms are *injective* (resp. *bijective*):  $\text{Mor}_{\mathcal{A}}(X, Y) \hookrightarrow \text{Mor}_{\mathcal{B}}(u(X), u(Y))$  (resp.  $\text{Mor}_{\mathcal{A}}(X, Y) \xrightarrow{\sim} \text{Mor}_{\mathcal{B}}(u(X), u(Y))$ ).

**Theorem 1.3.** Let  $\mathcal{C}$  be a category. Then we have *fully faithful functors*:

$$\begin{array}{ll} \mathcal{C} \longrightarrow \mathcal{F}un(\mathcal{C}^{\circ}, \text{Sets}) & \mathcal{C}^{\circ} \longrightarrow \mathcal{F}un(\mathcal{C}, \text{Sets}) \\ Y \longmapsto Y_{\bullet} & X \longmapsto X^{\bullet} \\ g \longmapsto g_{\bullet} & f \longmapsto f^{\bullet} \end{array}$$

*Proof.* We will only prove the first statement. The second follows by duality  $\mathcal{C}$  replaced by  $\mathcal{C}^{\circ}$ .

We have to show, for all  $X, Y \in \text{Ob } \mathcal{C}$ , the bijectivity of

$$\text{Mor}_{\mathcal{C}}(Y, X) \longrightarrow \text{Mor}(Y_{\bullet}, X_{\bullet})$$

In fact, this has been shown in the Yoneda Lemma 1.1 with  $u = X_{\bullet}$ .  $\square$

*Note.* The category  $\widehat{\mathcal{C}} = \mathcal{F}un(\mathcal{C}^{\circ}, \text{Sets})$  is called *category of presheaves of sets*. The theorem embeds  $\mathcal{C} \subset \widehat{\mathcal{C}}$  as a full subcategory.

**1.5. Adjoint functors.** Let  $\mathcal{A}, \mathcal{B}$  be categories and  $u : \mathcal{A} \rightarrow \mathcal{B}$  and  $v : \mathcal{B} \rightarrow \mathcal{A}$  be two functors. We now write  $\text{Hom}_{\mathcal{A}}, \text{Hom}_{\mathcal{B}}$  for the morphisms in the categories to distinguish them from the morphisms of functors  $\text{Mor}$ .

We consider the two functors  $\text{Hom}_{\mathcal{A}}(\cdot, v \cdot), \text{Hom}_{\mathcal{B}}(u \cdot, \cdot) : \mathcal{A} \times \mathcal{B} \rightarrow \text{Sets}$  and their morphisms as well as the two morphisms and their morphisms

$$u \circ v, id_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B} \qquad id_{\mathcal{A}}, v \circ u : \mathcal{A} \rightarrow \mathcal{A}$$

**Theorem 1.4.** *We have isomorphisms*

$$\begin{aligned} \text{Mor}(\text{Hom}_{\mathcal{A}}(\cdot, v \cdot), \text{Hom}_{\mathcal{B}}(u \cdot, \cdot)) &\xrightarrow{\sim} \text{Mor}(u \circ v, id_{\mathcal{B}}) \\ \varphi &\longmapsto \phi \\ \text{Mor}(\text{Hom}_{\mathcal{B}}(u \cdot, \cdot), \text{Hom}_{\mathcal{A}}(\cdot, v \cdot)) &\xrightarrow{\sim} \text{Mor}(id_{\mathcal{A}}, v \circ u) \\ \vartheta &\longmapsto \theta \end{aligned}$$

with  $\phi$  defined by  $\phi(B) = \varphi(v(B), B)id_{v(B)} : u(v(B)) \rightarrow B$  and correspondingly in the dual situation.

*Proof.* The proof is straightforward. If  $\phi : u \circ v \rightarrow id_{\mathcal{B}}$  is given we define

$$\begin{aligned} \varphi(A, B) : \text{Hom}_{\mathcal{A}}(A, v(B)) &\longrightarrow \text{Hom}_{\mathcal{B}}(u(A), B) \\ f &\longmapsto \phi(B) \circ u(f) \end{aligned}$$

The verification is trivial and left to the reader. See also [1, I.7, lemme 1, 2].  $\square$

Keeping the notation of theorem 1.4, let morphisms  $\varphi, \phi, \vartheta, \theta$  be given

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(\cdot, v \cdot) &\xrightarrow{\varphi} \text{Hom}_{\mathcal{B}}(u \cdot, \cdot) & \phi : u \circ v &\rightarrow id_{\mathcal{B}} \\ \text{Hom}_{\mathcal{A}}(\cdot, v \cdot) &\xleftarrow{\vartheta} \text{Hom}_{\mathcal{B}}(u \cdot, \cdot) & \theta : id_{\mathcal{A}} &\rightarrow v \circ u \end{aligned}$$

**Proposition 1.5.** *We have*

$$\begin{aligned} \vartheta \circ \varphi = id &\iff id_v : v \xrightarrow{\theta v} v u v \xrightarrow{v \phi} v \\ \varphi \circ \vartheta = id &\iff id_u : u \xrightarrow{u \theta} u v u \xrightarrow{\phi u} u \end{aligned}$$

*Proof.* Straightforward. See also [1, I.7, prop. 7]  $\square$

**Definition 1.6.** Under these conditions  $u$  is called *left adjoint* (or *co-adjoint*) to  $v$  and  $v$  is called *right adjoint* (or *adjoint*) to  $u$ .

The morphisms of functors  $\varphi$  resp.  $\vartheta$  are called *adjunction isomorphisms*, the morphisms  $\phi$  resp.  $\theta$  are called *adjunction morphisms*.

The categories  $\mathcal{A}$  and  $\mathcal{B}$  are called *equivalent* if the adjunction morphisms  $\phi$  and  $\theta$  are isomorphisms; the functors  $u$  and  $v$  are establishing the equivalency.

*Note.* Adjoints are determined up to isomorphy. An adjoint of an additive functor is additive. Adjoints are compatible with limits (e.g. kernels, products), i.e. they are left exact. Co-adjoints are compatible with co-limits (e.g. cokernels, coproducts (=sums)), i.e. they are right exact.

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