ζ–FUNCTION AND BERNOULLI NUMBERS

BERNDT E. SCHWERDTFEGER

Abstract. Basics on Dirichlet series and Riemann ζ–function.

Preface

This note is based on a manuscript written in 1971. It gathers the salient features of Dirichlet–series and their convergence, in particular the Riemann ζ–function, its *functional equation*[1] and some special values, including Bernoulli numbers.

The appendix treats the *analytical continuation* of the ζ–function in an elementary way, without using the *functional equation*.

Berlin, 28 February 2011

 v1.1, 2015-03-04

1. Dirichlet series

Lemma 1.1 (Abel's Lemma). *Let $(c_n)_{n\in\mathbf{N}}, (b_n)_{n\in\mathbf{N}}$ be sequences of complex numbers, let $C_n = \sum_{0\le\nu\le n} c_\nu$ be the n^{th} partial sum.*

Then we have

$$\sum_{n+1}^{n+p} c_\nu b_\nu = C_{n+p}b_{n+p} - C_n b_{n+1} + \sum_{\nu=n+1}^{n+p-1} C_\nu(b_\nu - b_{\nu+1}) \tag{1}$$

If moreover $|C_n| \le C$, $b_n \in \mathbf{R}$ is antitone, positive (i.e. $b_0 \ge b_1 \ge b_2 \ge \cdots \ge 0$), then we have

$$|\sum_{n+1}^{n+p} c_\nu b_\nu| \le 2Cb_{n+1}$$

Proof. $c_\nu = C_\nu - C_{\nu-1}$

$$\sum_{n+1}^{n+p} c_\nu b_\nu = \sum_{n+1}^{n+p} C_\nu b_\nu - \sum_{n+1}^{n+p} C_{\nu-1}b_\nu = \sum_{n+1}^{n+p} C_\nu b_\nu - \sum_{\nu=n}^{n+p-1} C_\nu b_{\nu+1}$$

which gives the formula (1), the upper bound follows from this (see also [1, V §2], [2, I Übung 16], [3, VIII §1], [6, VI §2]). □

Definition 1.1. A series like

$$f(s) = \sum_{n\ge1} \frac{a_n}{n^s}$$

is called a Dirichlet *series*.

2010 *Mathematics Subject Classification.* Primary 11M06; Secondary 11B68, 30B50.
Key words and phrases. ζ–function, Dirichlet series, Bernoulli numbers.

[1] added in 2011

Let $f_n(s)$ be the n^{th} partial sum. Traditionally the variable is written $s = \sigma + it$. Remark that $|n^s| = n^\sigma$

Theorem 1.2 (Convergence of DIRICHLET series)**.** *If $f(s)$ converges for one value $s = s_0$, then $f(s)$ converges for all s with $\sigma > \sigma_0$. More precisely:*
$f(s)$ converges uniformly on any compact subset of the open half plane $\sigma > \sigma_0$.

Such a compact set is contained in a compact set of the form $\sigma \geq \sigma_0 + \delta$, $|s - s_0| \leq R$ ($\delta > 0$, $R > 0$ suitably chosen).

Proof. We will apply ABEL's Lemma to

$$c_n = \frac{a_n}{n^{s_0}},\ b_n = \frac{1}{n^{s-s_0}}$$

We then have

$$c_n b_n = \frac{a_n}{n^s}$$

and

$$C_n = f_n(s_0)$$

and get from ABEL that

$$\begin{aligned} f_{n+p}(s) - f_n(s) = \sum_{\nu=n+1}^{n+p} \frac{a_\nu}{\nu^s} &= \frac{f_{n+p}(s_0)}{(n+p)^{s-s_0}} - \frac{f_n(s_0)}{(n+1)^{s-s_0}} \\ &\quad + \sum_{\nu=n+1}^{n+p-1} f_\nu(s_0)\Big(\frac{1}{\nu^{s-s_0}} - \frac{1}{(\nu+1)^{s-s_0}}\Big) \end{aligned}$$

As $f(s_0)$ converges, the partial sums are bounded

$$|f_n(s_0)| \leq M$$

Let now be $\sigma \geq \sigma_0 + \delta$, $|s - s_0| \leq R$ ($\delta > 0$, $R > 0$ arbitrary)

$$\frac{1}{\nu^{s-s_0}} - \frac{1}{(\nu+1)^{s-s_0}} = (s - s_0)\int_\nu^{\nu+1} \frac{dx}{x^{s-s_0+1}}$$

and therefore

$$\begin{aligned} |f_{n+p}(s) - f_n(s)| &\leq \frac{M}{(n+p)^\delta} + \frac{M}{(n+1)^\delta} + MR \cdot \sum_{\nu=n+1}^{n+p-1} \int_\nu^{\nu+1} \frac{dx}{x^{\delta+1}} \\ &\leq \frac{2M}{n^\delta} + \frac{MR}{\delta} \cdot \frac{1}{(n+1)^\delta} \\ &\leq (2 + \frac{R}{\delta})\frac{M}{n^\delta} \longrightarrow 0 \quad \text{with } n \longrightarrow \infty. \end{aligned}$$

□

The infimum of σ_0, such that $f(s)$ converges for $\sigma > \sigma_0$, is called the *convergence abscissa* and will be denoted $\sigma_0 = \sigma_0(f)$.

Obviously, f is holomorphic in the half plane of convergence.

We need another theorem for calculating the convergence abscissa.

Theorem 1.3. *Let $A_n = \sum_{\nu=1}^n a_\nu$.*

If $|A_n| \leq A \cdot n^{\sigma_1}$ for $n \gg 0$ (with suitable $\sigma_1 \geq 0$, $A > 0$), then $\sigma_0 \leq \sigma_1$.

In particular, for bounded A_n we have $\sigma_0 \leq 0$.

See [3, VIII §1], [6, VI prop. 8, 9].

Proof. We have ($c_n = a_n$, $b_n = n^{-s}$ in ABEL's Lemma)

$$f_{n+p}(s) - f_n(s) = \sum_{\nu=n+1}^{n+p} \frac{a_\nu}{\nu^s} = A_{n+p}(n+p)^{-s} - A_n(n+1)^{-s} + \sum_{\nu=n+1}^{n+p-1} A_\nu \big(\nu^{-s} - (\nu+1)^{-s}\big)$$

For s with $\sigma > \sigma_1 \geq 0$ (in particular $s \neq 0$) we have

$$\nu^{-s} - (\nu+1)^{-s} = s \cdot \int_\nu^{\nu+1} \frac{dx}{x^{s+1}}$$

For the absolute value we get

$$\begin{aligned}
|f_{n+p}(s) - f_n(s)| &\leq A \cdot (n+p)^{\sigma_1 - \sigma} + A \cdot (n+1)^{\sigma_1 - \sigma} \\
&+ \sum_{\nu=n+1}^{n+p-1} A \cdot \nu^{\sigma_1} |s| \cdot \int_\nu^{\nu+1} \frac{dx}{x^{\sigma+1}} \\
&\leq 2A \cdot n^{-(\sigma-\sigma_1)} + A \cdot |s| \sum_{\nu=n+1}^{n+p-1} \int_\nu^{\nu+1} \frac{dx}{x^{\sigma-\sigma_1+1}} \\
&\leq 2A \cdot n^{-(\sigma-\sigma_1)} + A \cdot |s| (\sigma - \sigma_1)^{-1} (n+1)^{-(\sigma-\sigma_1)} \\
&\leq \big(2 + \frac{|s|}{\sigma - \sigma_1}\big) \frac{A}{n^{\sigma-\sigma_1}} \longrightarrow 0 \quad \text{with } n \longrightarrow \infty.
\end{aligned}$$

We have shown that $f(s)$ is convergent for $\sigma > \sigma_1$, and therefore we must have $\sigma_1 \geq \sigma_0$. □

2. Riemann ζ–function

The RIEMANN *ζ–function* is the function to the DIRICHLET series $a_n = 1$:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

From the last theorem we see we can take $\sigma_1 = 1$ ($A_n = n$). As the *harmonic series* diverges, ζ has a *pole* at $s = 1$, therefore we have precisely $\sigma_0 = 1$.

Theorem 2.1 (analytical continuation)**.** *ζ can be analytically continued to the half plane $\sigma > 0$ as a meromorphic function with a single pole at $s = 1$. This pole is simple with residue $= 1$.*

Proof. Consider the alternating ζ_2–function

$$\zeta_2(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - + \dots$$

The sum of the coefficients is either 1 or 0, hence bounded and $\sigma_0 \leq 0$. On the other side it diverges for $s = 0$, hence exactly $\sigma_0 = 0$. If we add

$$\frac{2}{2^s}\zeta(s) = \frac{2}{2^s} + \frac{2}{4^s} + \dots$$

to the ζ_2–function we get the ζ–function:

$$\zeta_2(s) + 2^{-(s-1)}\zeta(s) = \zeta(s)$$

and we obtain

$$\zeta(s) = \frac{\zeta_2(s)}{1 - \dfrac{1}{2^{s-1}}}$$

yielding the meromorphic continuation in $\sigma > 0$.

Similarly, for $k = 2, 3, 4, \dots$ we let

$$\zeta_k(s) = 1 + \frac{1}{2^s} + \cdots + \frac{1}{(k-1)^s} - \frac{k-1}{k^s} + \frac{1}{(k+1)^s} + \cdots$$

This time the sum of the coefficients takes on the values $0, 1, 2, \dots, k-1$, and are bounded again and the same argument as above shows that $\sigma_0 = 0$.

$$\zeta(s) = \frac{\zeta_k(s)}{1 - \dfrac{1}{k^{s-1}}} \qquad k = 2, 3, 4, \dots \tag{2}$$

Poles of ζ can only occur, where the denominator in (2) vanishes, because the numerator is holomorphic (in the right half plane). This means for $k = 2, 3$ for example that

$$2^{s-1} = 1, \quad 3^{s-1} = 1$$

which necessarily implies that

$$s = 1 + \frac{2\pi i n}{\log(2)} = 1 + \frac{2\pi i m}{\log(3)}$$

which would give $2^m = 3^n$, hence $n = m = 0$. Therefore $s = 1$ is the only singularity.

We will finally show that the pole at $s = 1$ has the claimed properties: from the graph of $1/x^\sigma$ we can read that for $\sigma > 1$ we have

$$\frac{1}{\sigma - 1} = \int_1^\infty \frac{dx}{x^\sigma} \le \sum_{n \ge 1} \frac{1}{n^\sigma} = \zeta(\sigma) \le 1 + \int_1^\infty \frac{dx}{x^\sigma} = 1 + \frac{1}{\sigma - 1}$$

so we have $1 \le (\sigma - 1)\zeta(\sigma) \le \sigma$ and

$$\lim_{\sigma \to 1} (\sigma - 1)\zeta(\sigma) = 1 \tag{3}$$

If now $\zeta(s) = \sum_{-\infty}^{+\infty} a_n (s-1)^n$ is the LAURENT development around 1, we get from (3) that $a_n = 0$ for $n \le -2$ (simple pole) and $a_{-1} = 1$ (residue) $\square$

RIEMANN [4, VII, p. 147] makes use of the Γ–function to exhibit the *analytical continuation* of ζ to all of $\mathbf{C}$ and exposing its *functional equation* at the same time.

Theorem 2.2 (functional equation)**.**

$$\Gamma\Big(\frac{s}{2}\Big)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\Big(\frac{1-s}{2}\Big)\pi^{-\frac{1-s}{2}}\zeta(1-s)$$

Proof. We follow the reasoning of RIEMANN. He starts with

$$\frac{1}{n^s}\Gamma\Big(\frac{s}{2}\Big)\pi^{-\frac{s}{2}} = \int_0^\infty e^{-n^2\pi x} x^{\frac{s}{2}-1} dx$$

and introducing the *theta* series[2]

$$\psi(x) = \sum_1^\infty e^{-n^2\pi x}$$

summing up gives

$$\zeta(s)\Gamma\Big(\frac{s}{2}\Big)\pi^{-\frac{s}{2}} = \int_0^\infty \psi(x) x^{\frac{s}{2}-1} dx$$

[2]remark that $2\psi(x) + 1 = \vartheta(0, xi)$, see [5]

The function $g(t) = \exp(-t^2\pi x)$ has the FOURIER transform $\widehat{g}(t) = x^{-\frac{1}{2}} \exp(-t^2\pi/x)$. The POISSON formula

$$\sum g(n) = \sum \widehat{g}(n)$$

implies the *theta functional equation*

$$2\psi(x) + 1 = x^{-\frac{1}{2}}(2\psi(1/x) + 1).$$

Splitting the integral into $\int_1^\infty + \int_0^1$ and substituting this functional equation into the second integral he finally obtains

$$\begin{aligned}\zeta(s)\Gamma\big(\frac{s}{2}\big)\pi^{-\frac{s}{2}} &= \int_1^\infty \psi(x)x^{\frac{s}{2}-1}dx + \int_0^1 \psi(1/x)x^{\frac{s-3}{2}}dx + \\ &+ \frac{1}{2}\int_0^1 \big(x^{\frac{s-3}{2}} - x^{\frac{s}{2}-1}\big)dx = \\ &= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \psi(x)\big(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\big)dx\end{aligned}$$

which is invariant under $s \mapsto 1 - s$. □

See the appendix for another approach to *analytical continuation*.

3. BERNOULLI NUMBERS

See [1, V §8] p. 408, [6, VII §4] p. 147.

The numbers B_n defined in the development of the power series

$$\frac{x}{e^x - 1} = 1 + \sum_{n\geq 1} \frac{B_n}{n!}x^n \tag{4}$$

are called BERNOULLI–numbers. They are *rational*, as can be seen from the recursion formula (6) below.

For a polynomial

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

we will write symbolically

$$f(B) = a_0 + a_1B_1 + \cdots + a_nB_n$$

and similarly for power series. With this convention (4) can be re–written

$$e^{Bx} = \frac{x}{e^x - 1} \tag{5}$$

You see immediately by multiplying the power series that

$$e^{ax} \cdot e^{Bx} = e^{(a+B)x}$$

Theorem 3.1 (recursion formula for BERNOULLI numbers)**.**

$$(1 + B)^n - B^n = 0 \qquad n \geq 2. \tag{6}$$

Proof. From (5) it follows that

$$x = e^x \cdot e^{Bx} - e^{Bx} = e^{(1+B)x} - e^{Bx}$$

and you get the result by comparing the coefficients on both sides ! □

In particular (6) yields for $n = 2$: $B_1 = -\frac{1}{2}$. As is easy to see, the function

$$\frac{x}{e^x - 1} + \frac{x}{2} = 1 + \sum_{n\geq 2} B_n \frac{x^n}{n!}$$

is even, therefore $B_{2n+1} = 0$ $(n \geq 1)$. This is the reason why sometimes only the even BERNOULLI numbers are numerated. You find a table with the first 12 resp. 14 even BERNOULLI numbers in BOREWICZ–ŠAFAREVIČ [1] resp. SERRE [6] (the latter denotes our number B_{2n} by $(-1)^{n+1}B_n$).

Theorem 3.2. [1, V §8 Satz 6]

$$\zeta(2m) = (-1)^{m-1}\frac{(2\pi)^{2m}}{2\cdot(2m)!}B_{2m} \qquad m \geq 1 \tag{7}$$

Proof. A simple rearrangement of [2, V §2.3 (3.2)] p. 155 gives

$$\cot z = \frac{1}{z} + \sum_{n\geq 1} \frac{2z}{z^2 - (\pi n)^2}$$

As

$$\cot z = \frac{\cos z}{\sin z} = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = i + \frac{2i}{e^{2iz} - 1}$$

replacing $x = 2iz$ gives

$$\frac{x}{e^x - 1} = e^{Bx} = 1 - \frac{x}{2} + \sum_{n\geq 1}\frac{2x^2}{x^2 + (2\pi n)^2} \tag{8}$$

Now we have

$$\frac{x^2}{x^2 + (2\pi n)^2} = \sum_{m\geq 1}(-1)^{m-1}(\frac{x}{2\pi n})^{2m}$$

we put this into (8) and sum up over the n to obtain

$$e^{Bx} = 1 - \frac{x}{2} + \sum_{m\geq 1}(-1)^{m-1}\frac{2\cdot\zeta(2m)}{(2\pi)^{2m}}x^{2m}$$

and by comparison of coefficients this yields (7). □

The values of $\zeta(2n+1)$ are unknown. In 1978 APÉRY proved $\zeta(3) \notin \mathbf{Q}$.

Special values for $n \in \mathbf{N}, n > 0$ are:

$$\zeta(-2n) = 0 \qquad \text{('trivial' zeros)}$$
$$\zeta(1-2n) = -B_{2n}/2n \in \mathbf{Q} \qquad \text{(rational values)}$$

All non–trivial zeros are in the *critical strip* $0 \leq \sigma \leq 1$. RIEMANN found it most likely that they all lie on the *critical line* $\sigma = \frac{1}{2}$ (RIEMANN conjecture). This is known to be true for $\aleph_0$ zeros, but only some millions of them have explicitely been calculated.

APPENDIX A. ANALYTICAL CONTINUATION OF THE ζ–FUNCTION

Let

$$f_{k,n}(s) = \int_0^1 \frac{t^k}{(n+t)^{s+k}}dt \qquad \text{for } k \geq 0, n \geq 1$$

These functions are holomorphic everywhere in $\mathbf{C}$: as obviously

$$f_{0,n}(s) = \frac{1}{s-1}(\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}})$$

is holomorphic in $\mathbf{C}$, and we have

$$f_{k,n}(s) = \int_n^{n+1} \frac{(t-n)^k}{t^{s+k}} dt = \sum_{i=0}^{k} \binom{k}{i} (-n)^i f_{0,n}(s+i)$$

Since $|f_{k,n}(s)| \leq \dfrac{1}{n^{\sigma+k}}$, the sum

$$f_k(s) = \sum_{n=1}^{\infty} f_{k,n}(s)$$

converges for $\sigma > 1-k$ and is holomorphic in this half plane. In particular $f_0(s) = \dfrac{1}{s-1}$.

The function $\dfrac{1}{k+1} \cdot \dfrac{t^{k+1}}{(n+t)^{s+k}}$ has the derivative (in t):

$$\frac{t^k}{(n+t)^{s+k}} - \frac{s+k}{k+1} \cdot \frac{t^{k+1}}{(n+t)^{s+k+1}}$$

and by integration from 0 to 1 we get

$$f_{k,n}(s) - \frac{s+k}{k+1} f_{k+1,n}(s) = \frac{1}{k+1} \cdot \frac{1}{(n+1)^{s+k}}$$

and the summation from 1 to ∞ gives in the half plane $\sigma > 1-k$

$$f_k(s) - \frac{s+k}{k+1} f_{k+1}(s) = \frac{1}{k+1} \big(\zeta(s+k) - 1\big)$$

which implies by induction the formula valid for $\sigma > 1$:

$$1 + \frac{1}{s-1} - \zeta(s) = \sum_{i=1}^{k} \binom{s+i-1}{i} \frac{1}{i+1} \big(\zeta(s+i) - 1\big) + \binom{s+k}{k+1} f_{k+1}(s)$$

From this we conclude successively ($k = 0, 1, 2, \dots$) the holomorphy of the function on the left hand side in the whole s–plane.

References

[1] Senon I. Borewicz and Igor R. Šafarevič, *Zahlentheorie*, Birkhäuser, Basel, Stuttgart, 1966.

[2] Henri Cartan, *Elementare Theorie der Analytische Funktionen einer oder mehrerer komplexen Veränderlichen*, Hochschultaschenbücher, vol. 112, Bibliographisches Institut, Mannheim, Wien, Zürich, 1966.

[3] Serge Lang, *Algebraic Number Theory*, 2nd ed., Graduate Texts in Mathematics, vol. 110, Springer, 1994.

[4] Bernhard Riemann, *Gesammelte Mathematische Werke*, 2. Aufl., Teubner, Leipzig, 1892.

[5] Berndt E. Schwerdtfeger, *On theta functions* (2007), available at http://berndt-schwerdtfeger.de/wp-content/uploads/pdf/theta.pdf.

[6] Jean-Pierre Serre, *Cours d'arithmétique*, Presses Universitaires de France, Paris, 1970.