WAVE SPEED

BERNDT E. SCHWERDTFEGER

For Gerhard

Abstract. This article treats the *hull speed* of *displacement boats.*

Preface

Water is incompressible and offers a massive resistance for a boat hull. The buoyancy of a floating boat equals the weight of the fluid that it displaces (principle of Archimedes). It generates a wave system (bow and stern waves) that it cannot escape as long as it remains in water in its total length (so-called *displacement boat*). The maximal attainable limit speed of a *displacement boat* is called its *hull speed.* It is proportional to the square root of the length of the waterline of the boat.

In books on seamanship you find the following formula for the *hull speed*

$$v = R \cdot \sqrt{L} \qquad [2, \text{Rumpffahrt}], [3, \text{p. } 148, 159] \tag{1}$$

with the velocity v and the length of the waterline L. The proportionality factor is given by $R = 2.43$ if v is measured in `kn` (*knots*=nautical miles per hour) and L in `m` (*meters*). In this article I will explain the formula (1) and the factor R.

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1. Hull speed

We note some simple formulas, which result in (1) and explain the factor R. These formulas will be justified in section 2.

We consider a *wave* of length λ that propagates during the period T. The *velocity* c of the wave is

$$c = \frac{\lambda}{T} \qquad \text{(phase velocity)} \tag{2}$$

Instead of the period we consider its *frequency*

$$\omega = \frac{2\pi}{T} \qquad \text{(circular frequency)} \tag{3}$$

and its *wave number*

$$k = \frac{2\pi}{\lambda} \qquad \text{(circular wave number)} \tag{4}$$

This implies for c

$$c = \frac{\omega}{k} \tag{5}$$

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There is a connection between *frequency* and *wave number*: the *dispersion relation*

$$\omega^2 = kg \tag{6}$$

From (5) and (6) results

$$c^2 = \frac{g}{k} = \frac{g}{2\pi} \cdot \lambda \tag{7}$$

From (7) we conclude (1) and the meaning of the factor R is

$$R = \sqrt{\frac{g}{2\pi}} \tag{8}$$

which after rescaling to nautical miles and with the *gravity of Earth* $g = 9.81\ \mathtt{m}/\mathtt{s}^2$ yields the value given in the preface

$$R = \sqrt{\frac{9.81}{2\pi}} \cdot \frac{3.6}{1.852} = 2.42888$$

2. Hydrodynamics of gravity waves

The equations of motion of an incompressible fluid under the influence of *gravity* will be taken from hydrodynamics [1, §12. Schwerewellen].

Under the assumption $a \ll \lambda$, the *amplitude* a of the oscillation in the wave is *small* against the wave length λ, the *continuity equation* and the Euler *equation* simplify to [1, (12,4-5)]

$$\Delta\varphi = 0 \tag{9}$$

$$\left(\frac{\partial\varphi}{\partial z} + \frac{1}{g}\frac{\partial^2\varphi}{\partial t^2}\right)|_{z=0} = 0 \tag{10}$$

We consider the unbounded surface of an incompressible fluid, on which propagates a gravity wave in x-direction, homogenous in y; the gravitation acts in z-direction. The potential function φ we are looking for will be determined by a simply periodic function of time t and location x

$$\varphi = \cos(kx - \omega t) f(z)$$

ω is the *circular frequency* of the wave, k is the *wave number*, $\lambda = 2\pi/k$ the *wave length*, i.e. the period for the change of the current along the x-axis at a fixed point in time. From (9) we obtain the differential equation

$$f'' = k^2 f$$

and f lies in the two dimensional solution space

$$\mathbb{R}\cosh(kz) \oplus \mathbb{R}\sinh(kz) = \mathbb{R}\exp(kz) \oplus \mathbb{R}\exp(-kz).$$

We regard these conditions:

- the wave length is *small* against the depth of the fluid
- the depth of the fluid is $-h$

In the latter case as boundary condition on the bottom of the fluid the normal component of the velocity has to vanish

$$\frac{\partial\varphi}{\partial z}|_{z=-h} = 0 \quad \text{i.e. } f'(-h) = 0$$

This condition implies for $f(z) = ae^{kz} + be^{-kz}$ the relation $ae^{-kh} = be^{kh}$, hence that $f(z) = 2ae^{-kh}\cosh k(z+h)$.

In the case of *small* wave length against the depth of the fluid the solution evanescent in the depth (i.e. for $z \to -\infty$) is

$$\varphi = Ae^{kz} \cos(kx - \omega t)$$

In case of a fluid of depth $-h$ the solution is

$$\varphi = A \cos(kx - \omega t) \cosh k(z + h)$$

Feeding φ into the Euler equation (10) we get up to the factor $A \cos(kx - \omega t)$: $k - \omega^2/g = 0$ resp. $k \sinh kh - \omega^2 \cosh kh/g = 0$, hence finally

$$\omega^2 = kg \qquad \text{resp.} \tag{11}$$

$$\omega^2 = kg \tanh kh \tag{12}$$

This relation between the *frequency* ω and the *wave number* k is called *dispersion relation*. Of course, for $kh \to \infty$, i.e. if $\lambda \ll h$, the relation (12) tends to (11).

The velocity distribution in the moving fluid is the gradient of the potential function (case $\lambda \ll h$)

$$\nabla\varphi = Ake^{kz}(-\sin(kx - \omega t), 0, \cos(kx - \omega t))$$

The velocity is decreasing exponentially with increasing depth and the individual fluid particles describe nearly *circles* around their equilibrium (in case of finite depth the orbits are *ellipses*).

References

[1] Lev D. Landau and Jevgeni M. Lifschitz, *Hydrodynamik*, Lehrbuch der Theoretischen Physik, Verlag Harri Deutsch, Frankfurt am Main, 2007.

[2] Joachim Schult, *Segler Lexikon*, Delius Klasing, 2008.

[3] Deutscher Hochseesportverband Hansa (ed.), *Seemannschaft*, Delius Klasing, 2008.