ON EXPLICIT FORMULAS FOR DIFFERENTIAL FORMS IN THE THOM CLASS

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ABSTRACT. Construction of differential forms on vector bundles that represent the THOM cohomology class. These forms are used to calculate intersection numbers. Proof of GAUSS-BONNET formula.

INTRODUCTION

Starting point of this paper was a lemma by MILNOR [13], which expresses the intersection number $A \circ B$ of two transversal submanifolds A, B of complementary dimension in a manifold M by integration of differential forms, which can be associated canonically with the submanifolds of M. The formula that follows from MILNOR's Lemma reads

$$A \circ B = \int_M \tau_A \wedge \tau_B \qquad \text{see section 3.6}$$

The form $\tau_A \in \Omega^n(M)$, where $n = \operatorname{codim} A$, is basically a representative of the THOM class of the normal bundle of A in M. To exploit the formula above in a given situation you need to have exact knowledge about the forms τ_A. The purpose of this paper is to give explicit formulas for these differential forms.

First, to any homology class of M you can associate abstractly a cohomology class in the codimension. This is done in section 1.1 by the theory of HODGE and DE RHAM. Another canonical construction is given by the THOM class of the normal bundle of A in M, considered as a differential form in a tubular neighborhood. Both constructions coincide, as we will see in section 1.2.

So, you only need to perform the construction of a representative of the THOM class of a vector bundle E over M. By definition [8] the THOM class is a relative cohomology class in $H^n(E, E_0)$, and this cohomology is calculated by the complex $\Omega^*_{B \bmod S}$, the sheaf of differential forms on the ball bundle B of vectors of length ≤ 1 (relative to a fiber metric), which vanish on the boundary $S = \partial B$. This calculation of the relative cohomology is performed with standard methods of sheaf theory in section 0.2. The discussion in section 1.2 shows, that a representative of the THOM class $\tau \in \Omega^n(B, S)$ looks as follows:

$$\tau = \pi^*(\kappa) - d(h.\varphi^*(\Pi))$$

Here $\kappa \in \Omega^n(M)$ is a closed form, $\pi : B \longrightarrow M$ is the projection, $\Pi \in \Omega^{n-1}(S)$ is a form, which represents on all fibers the negative of the fundamental class, $\varphi : B_0 \longrightarrow S$ is the map, which maps any vector $e \neq 0$ to $e/\|e\| \in S$, and h is an auxilliary function that compensates for the poles of $\varphi^*(\Pi)$ in the zero section and guarantees the continuation into the zero section. As τ is a form on $B \bmod S$, the differential equation $d\Pi = \pi^*(\kappa)|S$ has to hold (which motivates the choice of the sign of Π). This reduces the problem to the construction of Π on the sphere bundle, such that Π satisfies the above conditions.

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For this Π you immediately think of a natural candidate: the fiber volume form. This fiber volume form can be constructed intrinsically with a Riemannian fiber metric and a metric connection (see [4]). Up to the factor $\mathrm{vol}(\mathbf{S}^{n-1})$ it represents on all fibers the fundamental class. To detect the form κ on the zero section, you have to calculate the exterior derivative and continue it into the zero section. The result is – as is to be expected – up to a factor the fiber volume form of the ball bundle, but – surprise ! – additionally also a curvature term, which comes from the curvature form of the covariant derivation of the connection on E. This curvature part is a "mixed" form, which measures some vertical (tangential to the fiber) as well as some horizontal (tangential to the base) directions. Also, the fiber volume form of the bundle is not closed in general. This natural candidate does not yield the right construction.

To find the right form Π we have to add a correcting term to the fiber volume form. This correcting term contains "mixed" forms and is to be build such that the derivative $d\Pi$ only measures horizontal directions. The construction of this form Π is due to CHERN [2] in the case of the tangential bundle and it was used for his proof of the GAUSS-BONNET formula. The generalization for arbitrary vector bundles is trivial and can be repeated along the same lines. This is done in section 3.4, as well as CHERN's proof of GAUSS-BONNET in section 3.5.

The section 2 develops the necessary theory of connections following Koszul [9], in a form suitable for our application. The first part of section 3 contains the construction of fiber forms on fiber bundles, a discussion of forms on vector bundles with poles in the zero section and some more technical preparations for the proof of the GAUSS-BONNET formula (index calculations). Eventually, in section 3.6, the definition of intersection numbers and their calculation by integration is given. There we also show, that the self intersection formula $M \circ M = \chi(M)$ is just another version of the GAUSS-BONNET formula.

NOTATIONS

In general we use the notation that is common in differential geometry (see [6], [7], [10]). As to bundle terminology: see HUSEMOLLER [8], as well as KOSZUL [9], DIEUDONNÉ [3]. As to sheaf terminology: see GODEMENT [5].

In particular we denote:

π_*	1. direct image of sheaves, 2. $= T\pi$ derivative of π = the tangential mapping
π^*	1. inverse image of modules (!), 2. pull back of differential forms 3. pulled back bundle (there is no confusion to be expected)
smooth	$\mathcal{C}^\infty$–differentiable
$\mathbb{R}_M$	sheaf of locally constant functions $M \longrightarrow \mathbb{R}$ (= associated sheaf to the constant pre-sheaf $\mathbb{R}$)

For the partial derivative of a smooth function $f : A \times B \longrightarrow C$ along A we introduce the following shortcut: let $f_b : A \longrightarrow C$ be the map $f_b(a) = f(a,b)$ (for $b \in B$). Its derivative $T_a(f_b) : T_aA \longrightarrow T_{f(a,b)}C$ is the partial derivative: $xb := T_a(f_b)(x)$ for a vector $x \in T_aA$

$$\begin{array}{ccc} T_aA \times B & \longrightarrow & TC \\ (x,b) & \longmapsto & xb \end{array}$$

and similarly for the derivative along B.

For the total derivative we have ($a \in A, b \in B$):

$$Tf = f_* : \quad \begin{array}{ccc} T_aA \times T_bB & \longrightarrow & T_{f(a,b)}C \\ (x,y) & \longmapsto & xy = xb + ay \end{array}$$

(see DIEUDONNÉ [3, 16.6.6]).

Manifolds are smooth, real and finite–dimensional, and also — if nothing is said to the contrary — without boundary.

Preface to the TeX Version

Version 1.0. This paper is an English translation of my *Diplomarbeit* (degree dissertation). I submitted it on October 31, 1972 to the Mathematics and Science Faculty of the University of Bonn, obtaining the diploma on November 21, 1972 from the dean Rolf Leis. Referees for the dissertation have been Günter Harder and Jacques Tits, other members of the board of examiners have been Pierre Gabriel (Pure Mathematics, Representation Theory), Rolf Leis (Applied Mathematics) and Wolfgang Priester (Astronomy).

Mainz, November 21, 2001 B. E. Schwerdtfeger

Version 1.1. There was a tiny change in numbering the formula (13) and (14) in section 2.2.2 (2002). The document is now under subversion control. No changes to the text were made.

Berlin, April 30, 2011 B. E. Schwerdtfeger

Contents

0. Preliminaries on bundles and modules

Manifolds (of class $\mathcal{C}^\infty$) are topological spaces, which are smoothly pasted together of euclidean spaces. Usually this is done with the help of an atlas of charts. It is at times useful — and this will often be done in this paper — to interpret a manifold as a geometric space $(M, \mathcal{O}_M)$ which is locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Here $\mathcal{O}_M$ indicates the structure sheaf of differentiable functions on M.

0.1. **Equivalence of vector bundles $/M$ and $\mathcal{O}_M$–modules.**

First let us recall the following facts (see [9]). Let $(M, \mathcal{O}_M)$ be a manifold.

0.1.1. *From vector bundles to $\mathcal{O}_M$–modules and back again.*

Theorem 0.1. *The category of vector bundles over M is equivalent to the category of locally free $\mathcal{O}_M$–modules. Fiber dimension and rank correspond to each other.*

Proof. The quasi–inverse functors are defined as follows:

To the bundle $\pi : E \longrightarrow M$ we associate the sheaf $\mathcal{E}$ of sections:

$\Gamma(U, \mathcal{E}) := \Gamma(U, E) = \{s : U \to E \mid \pi \circ s = id\}$.

In the other direction we define the *reduced fiber* of the $\mathcal{O}_M$–module $\mathcal{E}$ by

$$\mathcal{E}(x) := \mathcal{E}_x \otimes_{\mathcal{O}_{M,x}} \kappa(x) = \mathcal{E}_x / \mathfrak{m}_x \mathcal{E}_x$$

where $\kappa(x) = \mathbb{R}$ is the residue class field with $\mathcal{O}_{M,x}$–algebra structure

$$0 \to \mathfrak{m}_x \longrightarrow \mathcal{O}_{M,x} \longrightarrow \mathbb{R} \to 0$$

Finally we let $E := \coprod_{x \in M} \mathcal{E}(x)$.

The details as well as the functoriality of this construction is left to the reader. It is clear that the equivalence retains the exactness of sequences. $\square$

0.1.2. *Operations on bundles and modules.* The equivalence above is compatible with several common operations on bundles as well as modules, some of which we are going to mention:

	Vector bundles / M	locally free modules / M
direct sum	$E \oplus F$	$\mathcal{E} \oplus \mathcal{F}$
tensor product	$E \otimes F$	$\mathcal{E} \otimes \mathcal{F}$
homomorphisms	$HOM(E, F)$	$\mathcal{H}om(\mathcal{E}, \mathcal{F})$

We designate the Hom–functor in the category (VB/M) by $\mathrm{Hom}_M(E, F)$, that is

$$\mathrm{Hom}_M(E, F) = \Gamma(M, HOM(E, F)) = \Gamma(M, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$$

The resulting identifications will be used tacitly: for example a section in the dual bundle E^* will be interpreted as a fiberwise linear differentiable map $E \longrightarrow \mathbb{R}$.

0.1.3. *Bundle homomorphisms and global homomorphisms on sections.*

Lemma 0.2.

$$\mathrm{Hom}_M(E, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_M(M)}(\Gamma(M, E), \Gamma(M, F))$$

Proof. The map is defined like this: to any

$$\begin{array}{ccc} f : E & \longrightarrow & F \\ & \searrow \quad \swarrow & \\ & M & \end{array} \qquad \text{associate } s \longmapsto f \circ s$$

The inverse map is given by the following

$$\begin{array}{ccc} g : \Gamma(M,E) & \longrightarrow & \Gamma(M,F) \\ \downarrow & & \downarrow \\ \mathcal{E}_x & \dashrightarrow & \mathcal{F}_x \end{array}$$

the vertical maps are surjective, as $\mathcal{O}_M$–modules are soft. Let us show the maps on the stalks is well-defined: Let $s \in \Gamma(M,E)$ be such that $s_x = 0$, therefore $s|U = 0$ for a suitable open $U \subset M$, $x \in U$. Choose $\varphi \in \mathcal{O}_M(M)$ with

$$\varphi = \begin{cases} 0 & \text{in a neighbourhood } V \text{ of } x,\ V \subset U \\ 1 & \text{outside of } U \end{cases}$$

Because of $s = \varphi \cdot s$ and the $\mathcal{O}_M(M)$–linearity of g we have $g(s) = \varphi \cdot g(s)$, and this vanishes in the stalk $\mathcal{F}_x$.

The mapping on the stalks factors by a similar reasoning thru the reduced fiber and gives the searched for $f : E \longrightarrow F$. □

0.1.4. $\mathbb{R}$*–linear operators versus bundle morphisms.*

Note. This lemma allows to consider $\mathbb{R}$–linear maps

$$\Gamma(M,E) \longrightarrow \Gamma(M,F)$$

on the sections to be treated as homomorphisms of the bundles, if they are even $\mathcal{O}_M(M)$–linear !

Such maps on the global sections occur naturally as differential operators (like the covariant derivation), which are usually not $\mathcal{O}_M(M)$–linear.

0.1.5. *Behaviour under base change.*

Proposition 0.3. *Let a cartesian diagram of vector bundles be given*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ M' & \xrightarrow{f} & M \end{array}$$

that is $E' = M' \underset{M}{\times} E = f^*E$ *is the pulled back vector bundle. Let* $\mathcal{E}'$, $\mathcal{E}$ *be the sheaves of sections. Then we have*

$$\mathcal{E}' = f^*\mathcal{E}$$

(f^* denotes here the left adjoint to f_* in the category of modules).

First a

Lemma 0.4. *Let A be a local ring with residue class field k. Let P, Q be finitely generated A–modules. Let*

$$\varphi : P \longrightarrow Q$$

be A–linear and assume

$$\varphi \otimes k : P \otimes k \xrightarrow{\sim} Q \otimes k$$

is an isomorphism.

Then also $\varphi : P \xrightarrow{\sim} Q$ is an isomorphism.

Proof. By NAKAYAMA (applied to Ker and Coker). □

Proof of proposition 0.3. : The pull back of sections gives rise to the map

$$\begin{array}{ll} \varphi : \mathcal{E} \longrightarrow f_*\mathcal{E}' & \\ \varphi s(m') = (m', s \circ f(m')) & \quad s \in \mathcal{E}(U),\ m' \in M' \end{array}$$

We have to show that

$$\varphi^\sharp : f^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}'$$

is an isomorphism on the stalks, and by the lemma it is sufficient to show this for the reduced fibers, therefore for the fibers of the bundles. Now we have

$$(f^*\mathcal{E})(m') = (f^*\mathcal{E})_{m'} \otimes_{\mathcal{O}_{m'}} \kappa(m') = \mathcal{E}_{f(m')} \otimes_{\mathcal{O}_{f(m')}} \mathcal{O}_{m'} \otimes_{\mathcal{O}_{m'}} \kappa(m') = \mathcal{E}(f(m'))$$

and we will compare this fiber $E_{f(m')}$ with $E'_{m'} = \{m'\} \times E_{f(m')}$

We have the diagram

$$\begin{array}{ccc} \mathcal{E}_{f(m')} & \longrightarrow & \mathcal{E}'_{m'} \\ \downarrow & & \downarrow \\ E_{f(m')} & \longrightarrow & E'_{m'} \end{array}$$

and the lower map is given by $e \longmapsto (m', e)$, clearly an isomorphism. □

0.1.6. *Differential forms with values in vector bundles.*

Definition 0.1. The sections in the bundle $HOM(\bigwedge^k TM, E)$ are called differential forms of degree k (k–forms) with values in the vector bundle E.

The sheaf of sections is obviously $\Omega^k_M \otimes_{\mathcal{O}_M} \mathcal{E}$, which will also be abbreviated $\Omega^k_M \otimes E$. Because of the good functorial properties, especially compatibility with base change (pull back), this will not result in confusion.

0.1.7. *Pull back of morphisms.* We will define the *pull back* of these differential forms. In the situation with vector bundles $/P$ resp. $/M$

$$\begin{array}{ccccc} H & \xrightarrow{\sigma} & F & \qquad & E \\ \downarrow & & \downarrow & & \downarrow \\ P & \xrightarrow{\pi} & M & & M \end{array}$$

we have maps

$$\mathrm{Hom}_M(F, E) \longrightarrow \mathrm{Hom}_P(\pi^*F, \pi^*E) \longrightarrow \mathrm{Hom}_P(H, \pi^*E)$$

which come from the functor π^* *pull back of bundles* and induced by $H \longrightarrow \pi^*F$.

0.1.8. *Pull back of k–forms with values in vector bundles.* This will be applied to $H = \bigwedge^k TP$, $F = \bigwedge^k TM$, $\sigma = \bigwedge^k T\pi$ and we get the

Definition 0.2.

$$\pi^* : \mathrm{Hom}_M(\Lambda^k TM, E) \longrightarrow \mathrm{Hom}_P(\Lambda^k TP, \pi^* E)$$

Similarly at the sheaf level

$$\pi^* : \Omega^k_M \otimes E \longrightarrow \pi_*(\Omega^k_P \otimes \pi^* E)$$

The explicit formula for the pulled back form is

$$\pi^*(\omega)_p(x_1, \dots, x_k) = (p, \omega_{\pi(p)}(\pi_*(x_1), \dots, \pi_*(x_k))) \in \{p\} \times E_{\pi(p)}$$

where $p \in P$, $\quad x_1, \dots, x_k \in T_pP$ and ω is a k–form on M with values in E. This coincides with the usual definition for $E = M \times \mathbb{R}$, the scalar valued (normal) differential forms.

0.2. **Relative cohomology.** In this subsection M is a paracompact manifold and A a closed submanifold, let $i : A \hookrightarrow M$ be the inclusion.

0.2.1. *Definition of relative cohomology.*

Definition 0.3. The relative cohomology groups of A in M with real coefficients are

$$H^*(M, A; \mathbb{R}) := H^*(M; (\mathbb{R}_M)_{M-A})$$

By [5, 4.10.1] we have

$$H^*(M, A; \mathbb{R}) = H^*_{cl}(M - A; \mathbb{R})$$

and they coincide also with the relative cohomology à la Alexander–Spanier, according to the example in loc.cit. (the "*cl*" signifies that the support of the sections in $M - A$ have to be closed in M). We will show now, how this cohomology can be calculated by differential forms on M modulo A.

0.2.2. *A Lemma.*

Lemma 0.5. *The following sequence (see definition 0.2)*

$$\Omega^k_M \longrightarrow i_*\Omega^k_A \to 0$$

is exact as sequence of pre-sheaves.

Proof. Without restriction we will prove this for global sections. Choose a tubular neighbourhood U of A [10, chap. 4, §5] with retraction $r : U \longrightarrow A$, $A \subset U \subset M$. Let $f \in \mathcal{O}_M(M)$ be such that $f|A = 1$, $f|M - U = 0$ and let $\eta \in \Gamma(A, \Omega^k_A)$. Define a form $\omega \in \Gamma(M, \Omega^k_M)$ by $\omega|U := f \cdot r^*(\eta)$ and $\omega = 0$ outside the support of f. Obviously we have $i^*(\omega) = \eta$, qed. ☐

0.2.3. *Definition of forms on M modulo A.*

Definition 0.4. The sheaf of forms of M modulo A is defined as kernel in the sequence in Lemma 0.5

$$0 \to \Omega^k_{M \bmod A} \longrightarrow \Omega^k_M \longrightarrow i_*\Omega^k_A \to 0$$

0.2.4. *Calculation of relative cohomology by forms.*

Theorem 0.6.

$$H^*(M, A; \mathbb{R}) = H^*(\Gamma(\Omega^\bullet_{M \bmod A}))$$

Proof. By the theory of spectral sequences it suffices to show, that the complex $\Omega^\bullet_{M \bmod A}$ constitutes a cohomological trivial resolution of the sheaf $(\mathbb{R}_M)_{M-A}$. We have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{R}_M)_{M-A} & \longrightarrow & \mathbb{R}_M & \longrightarrow & (\mathbb{R}_M)_A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^\bullet_{M \bmod A} & \longrightarrow & \Omega^\bullet_M & \longrightarrow & i_*\Omega^\bullet_A & \longrightarrow & 0 \end{array}$$

with exact rows (the first one by [5, th. 2.9.3]).

As A is closed, we have $(\mathbb{R}_M)_A = i_* i^* \mathbb{R}_M = i_* \mathbb{R}_A$. Furthermore i_* is exact. By the Poincaré Lemma the last two vertical arrows are resolutions, and therefore also the first arrow. By [5, 4.9.1 (a)] we have

$$H^q(M; i_*\Omega^\bullet_A) = H^q(A; \Omega^\bullet_A)$$

and this group vanishes for $q \geq 1$, as the $\Omega^\bullet_A$ are soft. As well we have $H^q(M; \Omega^\bullet_M) = 0$ for $q \geq 1$. It is clear that the $\Omega^\bullet_{M \bmod A}$ are not soft in general. Nevertheless we have here as well:

$$H^q(M; \Omega^\bullet_{M \bmod A}) = 0 \qquad \text{for } q \geq 1$$

For $q \geq 2$ this follows from the long exact cohomology sequence of the above diagram. For $q = 1$ this is clear by Lemma 0.5, qed. □

0.2.5. *Relative Cohomology with family of supports.*

Note. The same proof also shows that

$$H^q_\Phi(M, A; \mathbb{R}) = H^q(\Gamma_\Phi(\Omega^\bullet_{M \bmod A}))$$

Furthermore $H^*(M, A; \mathbb{R}) = H^*_{cl}(M - A; \mathbb{R})$ in 0.2.1 is induced by the natural map $\Gamma(M, \Omega^\bullet_{M \bmod A}) \to \Gamma_{cl}(M - A, \Omega^\bullet_M)$ (recall, the "cl" signifies that the support of the sections in $M - A$ have to be closed in M).

0.3. **Relative differentials and orientation of submersions.**

0.3.1. *Bundle of vertical vectors.* Let $\pi : P \longrightarrow M$ be smooth and its derivation $\pi_* : TP \longrightarrow \pi^*TM$ be locally of constant rank (a *subimmersion*). Then the kernel bundle exists, its elements are called the *vertical* vectors. The bundle of these vertical vectors will be denoted V.

If $p \in P$ is above $m \in M$ and if $P_m = \pi^{-1}(m)$ is the fiber of π (it is a closed submanifold of P), then we have

$$V_p = T_p(P_m)$$

i.e. the vertical vectors are the vectors tangential to the fiber ([3, 16.8.8]).

0.3.2. *Bundle of relative forms.* The sheaf of sections of the dual bundle V^* is denoted by $\Omega^1_{P/M}$ and the *sheaf of relative k–forms of P over M* is

Definition 0.5. $\Omega^k_{P/M} := \bigwedge^k \Omega^1_{P/M}$

By definition we have the following exact sequences.

$$0 \to V \longrightarrow TP \longrightarrow \pi^* TM \tag{1}$$

$$\pi^* T^* M \longrightarrow T^* P \longrightarrow V^* \to 0 \tag{2}$$

$$\pi^* \Omega^1_M \longrightarrow \Omega^1_P \longrightarrow \Omega^1_{P/M} \to 0 \tag{3}$$

By theorem 0.1 (3) is left injective exactly when (1) is right surjective, that is π is *submersive*. For the inclusion of the fiber $j_m : P_m \hookrightarrow P$ we have by 0.3.1

$$j^*_m(\Omega^k_{P/M}) = \Omega^k_{P_m} \tag{4}$$

0.3.3. *The vertical bundle of a vector bundle.*

Example 0.1. If $\pi : P = E \longrightarrow M$ is a vector bundle, than for $e \in E_m$ $(\pi(e) = m)$ $V_e = T_e E_m = \{e\} \times E_m$ and $V = \pi^* E = E \times_M E$ is the vertical bundle. $\Omega^1_{E/M} = \pi^* \mathcal{E}^\vee$ where $\mathcal{E}$ is the sheaf of sections to $E \to M$. Let $n = \dim E_m\ \forall m \in M$ be the fiber dimension, then the vector bundle is orientable if $\bigwedge^n \mathcal{E}$ and therefore $\bigwedge^n \mathcal{E}^\vee$ is a free $\mathcal{O}_M$–module. Then also $\Omega^n_{E/M} = \pi^* \bigwedge^n \mathcal{E}^\vee$ is a free $\mathcal{O}_E$–module.

0.3.4. *Orientation on fiber bundles.*

Definition 0.6. A surjective submersion $\pi : P \longrightarrow M$ of (common) fiber dimension n is called *orientable*, iff $\Omega^n_{P/M}$ is a free $\mathcal{O}_P$–module. An *orientation* of π is a chosen isomorphism $\mathcal{O}_P \xrightarrow{\sim} \Omega^n_{P/M}$.

Note. Because of (4) the orientability of π implies the orientability of all fibers.

0.3.5. *Factoring the highest degree forms into base and fiber.* Let $d = \dim M$. The exact sequence

$$0 \to \pi^* \Omega^1_M \longrightarrow \Omega^1_P \longrightarrow \Omega^1_{P/M} \to 0 \tag{5}$$

gives rise to a canonical isomorphism

$$\pi^* \Omega^d_M \otimes_{\mathcal{O}_P} \Omega^n_{P/M} \xrightarrow{\sim} \Omega^{d+n}_P \tag{6}$$

which is given explicitly as follows: you define

$$\begin{array}{ccccc} \Omega^d_M & \otimes & \pi_* \Omega^n_{P/M} & \longrightarrow & \pi_* \Omega^{d+n}_P \\ \alpha & \otimes & \omega & \longmapsto & \pi^* \alpha \wedge \omega \end{array}$$

The form on the right makes sense, as applying vectors for a relative form is only defined for vertical vectors, but those vanish for a pulled back form. By adjunction rules of π^*, π_* we obtain a morphism

$$\pi^*(\Omega^d_M \otimes \pi_* \Omega^n_{P/M}) \longrightarrow \Omega^{d+n}_P$$

which factors thru (6). Isomorphy is checked locally, where it is clear.

0.3.6. *Orientability in fibrations.*

Proposition 0.7. (1) *P is orientable $\Rightarrow$ $\forall m \in M$ P_m is orientable*
(2) *M and π are orientable $\Rightarrow$ P is orientable*
(3) *M and P are orientable $\Rightarrow$ π is orientable*

Proof. This follows from (6). □

To the contrary, M is in general not orientable, even if P and π are. If P and M are oriented, then an orientation of π will be fixed by (6) (that is: first the base, then the fiber).

0.3.7. *Integration along the fiber of a fibration.* In the following let $\pi : P \longrightarrow M$ be a locally trivial fibration with typical fiber F with boundary of dimension n. Furthermore let π be oriented (and therefore F orientable). The "!" in the sequel shall mean: *compact support.*

Proposition 0.8. *There exists a distinguished $\mathbb{R}$–linear mapping*

$$\int_\pi : \Gamma_!(P, \Omega_P^{k+n}) \longrightarrow \Gamma_!(M, \Omega_M^k) \qquad 0 \le k \le d = \dim M$$

with the properties: for any $\alpha \in \Gamma(M, \Omega_M^r)$, $\omega \in \Gamma_!(P, \Omega_P^{k+n})$ we have

$$\int_\pi \pi^*\alpha \wedge \omega = \alpha \wedge \int_P \omega \tag{7}$$

$$\int_{\pi'} \varphi^*(\omega) = \int_\pi \omega \tag{8}$$

where $\varphi : P' \overset{\sim}{\longrightarrow} P \;/\; M$ is a diffeomorphism of oriented submersions.

Furthermore we have for the support of the forms that

$$\operatorname{supp} \int_\pi \omega \subset \pi(\operatorname{supp} \omega) \tag{9}$$

Theorem 0.9 (Fubini)**.** *If P and M are oriented, then we have for $k = d$ that*

$$\int_P = \int_M \circ \int_\pi$$

or otherwise put, the following diagram is commutative

$$\begin{array}{ccc} \Gamma_!(P, \Omega_P^{d+n}) & \xrightarrow{\int_\pi} & \Gamma_!(M, \Omega_M^d) \\ & {\scriptstyle \int_P} \searrow \quad \swarrow {\scriptstyle \int_M} & \\ & \mathbb{R} & \end{array}$$

0.3.8. *Construction of $\int_\pi$.* With the help of a partition of 1 the support can be chosen suitably small, and it suffices to define $\int_\pi$ locally. Because of (8) we may thus assume P to be globally trivial.

Let $P = M \times F$, with F oriented.

Now we have $\Omega_{P/M}^n = p_2^*\Omega_F^n$, where $p_2 : P \longrightarrow F$ is the projection, $\Omega_P^1 = \pi^*\Omega_M^1 \oplus \Omega_{P/M}^1$, i.e. the sequence (5) is split–exact.

By the map $\Omega_P^{k+n} = \coprod_{r+s=k+n} \pi^*\Omega_M^r \otimes \Omega_{P/M}^s \longrightarrow \pi^*\Omega_M^k \otimes \Omega_{P/M}^n$ we associate to any $k+n$ form on P a $\pi^* \bigwedge^k T^*M$–valued relative n–form

$$\begin{array}{ccc} \Gamma(P, \Omega_P^{k+n}) & \longrightarrow & \Gamma(P, \Omega_{P/M}^n \otimes \pi^* \bigwedge^k T^*M) \\ \omega & \longmapsto & \overline{\omega} \end{array}$$

The pulled back n–form $j_m^*(\overline{\omega})$ on the fiber P_m has values in the vector space $j_m^*\pi^* \bigwedge^k T^*M = \bigwedge^k T_m^*M$ and can be integrated

$$(\int_\pi \omega)(m) := \int_{P_m} j_m^*(\overline{\omega})$$

We have thus defined a section $\int_\pi \omega : M \longrightarrow \bigwedge^k T^*M$. In a local chart we can read off that this section is smooth: Let without restriction be $M = \mathbb{R}^d$, $F = \mathbb{R}^n_+$ (upper halfplane) and $x = (x^1, \dots, x^d)$, $y = (y^1, \dots, y^n)$ coordinates. With $I_r = \{\mu \in \mathbb{N}^r \mid 1 \leq \mu_1 < \cdots < \mu_r \leq d\}$, $J_s = \{\nu \in \mathbb{N}^s \mid 1 \leq \nu_1 < \cdots < \nu_s \leq n\}$ and the abbreviation $dx^\mu = dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$ the form ω is written

$$\omega = \sum_{r+s=k+n} \sum_{\mu \in I_r, \nu \in J_s} f_{\mu\nu} dx^\mu \wedge dy^\nu$$

and we have

$$\int_\pi \omega = \sum_{\mu \in I_k} (\int_F f_{\mu 1..n} dy^{1..n}) dx^\mu$$

and $\int_\pi \omega$ is smooth (partial differentiation under the integral).

The formulas (7)–(9), as well as the theorem of Fubini, follow from this local description.

1. The Thom class via de Rham–Hodge theory

1.1. **Cohomology classes associated to cycles.**

From now on we assume M to be a compact, oriented manifold. The Poincaré *duality* maps a cycle to a cohomology class in its codimension. A simple description of this mapping is given by the de Rham–Hodge *theory* (see [14]), which will be sketched now.

1.1.1. de Rham *isomorphism.* As we exclusively consider cohomology groups with coefficients $\mathbb{R}$, we will only write $H^*(M)$. The cohomology classes are represented by closed differential forms, see section 0.2.4. Analogously the singular homology classes in $H_*(M)$ are represented by differentiable cycles. The integral over a closed differential form ω along a cycle c, $\int_c \omega$, depends only of the co- resp. homology class (because of the theorem of Stokes). The resulting bilinear map

$$\begin{array}{ccccc} H_q(M) & \times & H^q(M) & \longrightarrow & \mathbb{R} \\ (c & , & \omega) & \longmapsto & \int_c \omega \end{array}$$

is non–degenerate: this is the theorem of de Rham, or equivalently (recall, that the cohomology is finite), that

$$H^q(M) \xrightarrow{\sim} (H_q(M))^*$$

is an isomorphism ([14, 4.17]).

1.1.2. HODGE *theory.* Let $n = \dim M$. The integral $\int_M \omega \wedge \eta$, where ω, η are closed differential forms of complementary degree, depends only on their cohomology classes (again by STOKES theorem). It is an easy corollary of HODGE *theory* (see [14, 6.13]), that the following bilinear mapping is non–degenerate

$$\begin{array}{ccccc} H^{n-q}(M) & \times & H^q(M) & \longrightarrow & \mathbb{R} \\ (\omega & , & \eta) & \longmapsto & \int_M \omega \wedge \eta \end{array}$$

and this gives us an isomorphism

$$H^q(M) \xrightarrow{\sim} (H^{n-q}(M))^*$$

(beware the order !).

1.1.3. POINCARÉ *duality.* Above isomorphisms give by composition of HODGE 1.1.2 with DE RHAM 1.1.1

$$H^q(M) \xrightarrow{\sim} (H^{n-q}(M))^* \xrightarrow{\sim} H_{n-q}(M)$$

the POINCARÉ *duality*

$$D : H^q(M) \xrightarrow{\sim} H_{n-q}(M)$$

This coincides with the convention [8, 17.5.4]: the cup–cap relation in [8, 17.5.1] reads

$$(*) \qquad \int_M \omega \wedge \eta = \int_{D\eta} \omega$$

and this is exactly the definition of the POINCARÉ duality map D.

The inverse $\gamma = D^{-1}$ associates to each cycle a cohomology class

$$(**) \qquad \gamma : H_q(M) \xrightarrow{\sim} H^{n-q}(M)$$

In particular we have associated to each closed oriented submanifold $A \subset M$ of dimension k a cohomology class $\gamma_A \in H^{n-k}(M)$: we map the fundamental cycle of A in $H_k(A)$ via the inclusion in $H_k(M)$ and apply $(**)$. By the cup–cap–formula $(*)$ γ_A is uniquely defined by the property, that for all k–forms $\omega \in \Gamma(M, \Omega^k_M)$ with $d\omega = 0$ we have

$$(10) \qquad \int_M \omega \wedge \gamma_A = \int_A \omega$$

In the next section we give a geometric interpretation of the classes γ_A. Let us now first fix the special cases $A = M$ and $A = \{m\}$ point:

(1) The associated cohomology class of M is the constant function 1, $\gamma_M = 1$.
(2) To each point corresponds the same cohomology class, the fundamental class $\mu = [M] \in H^n(M)$

$$\forall\, m \in M \quad \gamma_{\{m\}} = \mu$$

Proof. Ad (1): The equation (10) reads here $\int_M \omega = \int_M \omega \cdot \gamma_M$, therefore $\gamma_M = 1$.

Ad (2): (10) reads with $\omega = 1$: $1 = \int_{\{m\}} \omega = \int_M \gamma_{\{m\}}$, which characterizes the fundamental class. □

1.2. **The THOM class.** The associated cohomology class γ_A of a submanifold $A \subset M$ is essentially the THOM class of the normal bundle of A in M. We therefore discuss the THOM class of an arbitrary oriented vector bundle and derive an important formula for a representing differential form in this class.

1.2.1. THOM *class of vector bundle.* Let $E \longrightarrow M$ be an oriented vector bundle, let E_0 be the vectors $\neq 0$, let n be the fiber dimension and $j_m : E_m \hookrightarrow E$ be the inclusion of the fiber.

By definition the THOM *class* of E

$$U_E \in H^n(E, E_0) \qquad \text{(real coefficients)}$$

is uniquely determined by the property that it induces on each fiber the fundamental class, or, put otherwise: under the orientation map (integration along the fiber) the class $j_m^*(U_E)$ is mapped onto 1:

$$\int_{E_m} : H^n(E_m, E_m - \{0\}) \longrightarrow \mathbb{R}, \qquad \int_{E_m} U_E = 1$$

For this and additional formal properties of the THOM class (THOM isomorphism, multiplicativity) compare [8, 16.7,8,10].

We will give in the sequel the interpretation in the framework of DE RHAM theory.

1.2.2. *Differential forms in the* THOM *class.* Now let additionally a Riemannian fiber metric g be given on E, let B denote the vectors of length ≤ 1, $S = \partial B$ those of length $= 1$. $\pi : B \longrightarrow M$ is an oriented ball bundle, $\pi_1 : S \longrightarrow M$ is an oriented sphere bundle. By well known theorems of algebraic topology (excision theorem) we can cut off the open set $E - B$ from E without changing the cohomology groups. Furthermore, $B_0 = E_0 \cap B$ is homotopy equivalent to S (as $B_0 \simeq S \times (0, 1]$ and $(0, 1]$ is contractable). Hence $H^*(E, E_0) \xrightarrow{\sim} H^*(B, B_0) \xrightarrow{\sim} H^*(B, S)$. This last group can be calculated by relative differential forms of B modulo S according to theorem 0.6. With this 1.2.1 can be reformulated like this

Proposition 1.1. *There is a differential form*

$$\tau_E \in \Omega^n(B, S) := \Gamma(B, \Omega^n_{B \bmod S})$$

with the properties

$$\int_{B_m} \tau_E = 1, \qquad d\tau_E = 0$$

and its cohomology class, the THOM *class, is uniquely defined by these properties.*

1.2.3. *Structure of differential* THOM *forms.* We will now derive the announced formula for τ_E. To this end, we will decompose the form into one part that comes from the zero section, and into another outside of the zero section, which is a coboundary.

Let j_m be the inclusion of the fiber over $m \in M$ for the ball as well as for the sphere bundle, $i : S \hookrightarrow B$, $r : B_0 \longrightarrow \mathbb{R}$, $r(e) = \|e\| = g(e, e)^{1/2}$ the "radius", $\varphi : B_0 \longrightarrow S$, $\varphi(e) = \frac{1}{\|e\|} e$ the "angle", $\nu : M \longrightarrow B$, $\nu(m) = 0_m \in B_m$ the zero section.

Now the π, ν establish a homotopy equivalence of M and B, as $\pi \circ \nu = 1_M$, $\nu \circ \pi \sim 1_B$ (linearly homotopic in each fiber).

A piece of the long exact cohomology sequence given by 0.2 gives the following diagram

$$\begin{array}{ccccc} H^n(B,S) & \longrightarrow & H^n(B) & \xrightarrow{i^*} & H^n(S) \\ & & \nu^* \downarrow \uparrow \pi^* & & \\ & & H^n(M) & & \end{array}$$

We define the form induced by $\tau = \tau_E$ on the zero section by

$$\kappa := \nu^*(\tau) \in \Omega^n(M)$$

It is closed, as τ is: $d\kappa = 0$.

From the diagram above we conclude that $\pi^*(\kappa)$ represents the same cohomology class in $H^n(B)$, as the image of τ; these forms differ therefore by a coboundary:

$$(*) \qquad \exists\, \alpha \in \Omega^{n-1}(B) \quad \text{with} \quad \pi^*(\kappa) = \tau + d\alpha$$

When we induce this relation on the zero section, we get $\nu^*(d\alpha) = d\nu^*(\alpha) = 0$. The form α will in general not be closed itself. In fact, we only need the knowledge of α on the boundary. Since the additional condition on τ, to represent the fundamental class on the fibers, gives:

$$\int_{B_m} j_m^*\tau = -\int_{B_m} j_m^* d\alpha = -\int_{B_m} dj_m^*\alpha = -\int_{S_m} j_m^* i^*\alpha = 1$$

We are led to define

$$\Pi := i^*\alpha \in \Omega^{n-1}(S)$$

From $(*)$ we obtain the equation (as $i^*\tau = 0$):

$$(**) \qquad \pi_1^*(\kappa) = d\Pi$$

What happens outside the zero section ? By pull back to B_0 via $\varphi : B_0 \longrightarrow S$ $(**)$ yields ($\pi_0 := \pi|B_0$):

$$\pi_0^*(\kappa) = d\varphi^*\Pi$$

It is clear now, how to proceed: continue the form $d\varphi^*\Pi$ into the zero section and then to take $\tau = \pi^*(\kappa) - d\varphi^*\Pi$. To do this, we have to get rid of the pole, that φ has in the zero section. The phenomenon of the occurence of poles will be explained in 1.2.5 for the trivial case of M =point, and we will conduct the discussion in 3.2 in the general case. It will become evident, that $\varphi^*\Pi$ has a pole of order $n-1$. To avoid these poles, we choose a suitable auxilliary function, that vanishes near the zero section.

This investigation is summarized in the

1.2.4. *Structure Theorem.*

Theorem 1.2. *Let $\kappa \in \Omega^n(M)$, $\Pi \in \Omega^{n-1}(S)$ be forms which satisfy*

$$(C) \qquad d\kappa = 0, \quad d\Pi = \pi_1^*(\kappa), \quad \int_{S_m} \Pi = -1$$

Let $h : B \longrightarrow \mathbb{R}$ be smooth with

$$h = \begin{cases} 0 & \textit{inside of a } \delta\textit{–neighbourhood of the zero section} \\ 1 & \textit{outside of an } \varepsilon\textit{–neighbourhood of the zero section} \end{cases}$$

where $0 < \delta < \varepsilon < 1$.

Then the form in $\Omega^n(B, S)$

$$\tau = \pi^*(\kappa) - d(h \cdot \varphi^*(\Pi))$$

is a representative of the Thom *class.*

1.2.5. *The trivial case $M = point$.* In this case we have $E = \mathbb{R}^n$, coordinates will be $x^1, \dots, x^n$. The volume form of E is $v = dx^1 \wedge \dots \wedge dx^n$, the volume form of $S = \mathbb{S}^{n-1}$ is

$$\sigma = \sum_{i=1}^{n} (-1)^{i-1} x^i \cdot dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

(“ $\widehat{\ }$ ” signifies as usual an absent term).

We have $d\sigma = n \cdot v$. The function $\varphi : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{S}^{n-1}$ has the coordinates $\varphi^i = x^i \circ \varphi = \varphi^*(x^i) = \frac{x^i}{r}$, its differentials are

$$d\varphi^i = \frac{dx^i}{r} - \frac{x^i}{r^2} dr$$

As $r^2 = \sum_i (x^i)^2$ we have the relation $rdr = \sum_i x^i dx^i$ and the multiplication with σ yields $rdr \wedge \sigma = r^2 \cdot v$ and therefore $dr \wedge \frac{\sigma}{r} = v$. As dr is the unit normal form, the form $*dr = \frac{\sigma}{r}$ is the volume form of any sphere around 0. By the above formulas we get $\varphi^*(\sigma) = \frac{\sigma}{r^n} + T$, where T contains the radial terms; but as σ as well as $\varphi^*(\sigma)$ vanishes in radial direction, we have in fact $T = 0$ (which, of course, can be calculated directly as well). The form $\frac{\sigma}{r^n}$ has a pole of order $n-1$ in the origin:

Let $e_1, \dots, e_{n-1}$ be positively oriented, orthonormal tangential vectors at the sphere $\mathbb{S}^{n-1}(\varepsilon)$ of radius $\varepsilon > 0$, then we have $\frac{\sigma}{r^n}(e_1, \dots, e_{n-1}) = \frac{1}{\varepsilon^{n-1}}$. This form is closed: $d\frac{\sigma}{r^n} = -\frac{n}{r^{n+1}} dr \wedge \sigma + \frac{1}{r^n} d\sigma = \frac{1}{r^n}(-n \cdot dr \wedge \frac{\sigma}{r} + n \cdot v) = 0$. As $\int_{\mathbb{S}^{n-1}} \sigma = \mathrm{vol}(\mathbb{S}^{n-1})$, the system of equations (C) in theorem 1.2 is solved by $\kappa = 0$, $\Pi = -\frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})}\sigma$ (the form τ is a boundary, but <u>not</u> a boundary modulo $\mathbb{S}^{n-1}$!).

1.2.6. *The cohomology class of a cycle is the* Thom *class.* We will now show that the associated cohomology class γ_A is the Thom class of A in M.

Let M be compact, oriented, $A \subset M$ closed, oriented of codimension n. Then the the normal bundle of A in M, $E \longrightarrow A$ is oriented as well (see 0.3.6). We have the situation

$$\begin{array}{ccc} TA \oplus E & \hookrightarrow & TM \\ \downarrow & & \downarrow \\ A & \hookrightarrow & M \end{array}$$

Choose a Riemannian metric on M such that the exponential map restricted to the ball bundle $B \subset E$ is a diffeomorphism onto a closed submanifold (with boundary) of M (a so called tubular neighbourhood). This is always possible, since after multiplying the metric by a scalar factor > 0 the geodesics in M do not change, and therefore so doesn't exp, but the lengths of the vectors do. Of course E gets the induced fiber metric. We have now

$$\begin{array}{ccc} B & & \\ {\scriptstyle\pi}\downarrow & \overset{\exp}{\searrow} & \\ A & \hookrightarrow & B' \subset M \end{array}$$

The form $\tau_E \in \Omega^n(B, S)$ can by the theorem 1.2 be chosen such that the support is *distant from the boundary.* Therefore the transported form via exp to B' can be smoothly continued on all of M by zero. This form will be denoted τ_A:

$$\tau_A := (\exp^{-1})^*(\tau_E) \in \Omega^n(M)$$

Theorem 1.3. *τ_A lies in the cohomology class γ_A.*

Proof. By (10) in section 1.1.3 we have to show: for any closed form ω in M of degree $\dim A$ we have

$$\int_A \omega = \int_M \omega \wedge \tau_A$$

As the support of τ_A lies in B', it suffices to show this formula for forms ω with support in B'. It is clear that $\exp^* \omega - \pi^* i^* \omega$ is exact, as it vanishes on the zero section of A, see section 1.2.3; i.e. $\exp^* \omega$ and $\pi^* i^* \omega$ are cohomolog. Then we have

$$\int_M \omega \wedge \tau_A = \int_{B'} \omega \wedge \tau_A = \int_B \exp^* \omega \wedge \tau_E = \int_B \pi^* i^* \omega \wedge \tau_E = \int_A \circ \int_\pi \pi^* i^* \omega \wedge \tau_E$$

by Fubini theorem 0.9 and by (7) in prop. 0.8 this

$$= \int_A i^* \omega \cdot \int_\pi \tau_E$$

but $\int_\pi \tau_E = 1$ (constant function 1 on A) by definition of the fiber integral and the properties of τ_E. □

1.2.7. *The differential equation for the* Thom *forms.*

Note. The representing formula of theorem 1.2 will only be of value, when we know that the differential and integral equations can always be solved. This will be shown in section 3.

2. Connection

The theory presented here follows Koszul [9].

2.1. **Covariant derivation.**

2.1.1. *Definition of a covariant derivation.* Let $(M, \mathcal{O}_M)$ be a manifold, $\mathcal{E}$ a locally free $\mathcal{O}_M$–module.

Definition 2.1. A *covariant derivation* for $\mathcal{E}$ is an $\mathbb{R}$–linear mapping of sheaves

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_M \otimes_{\mathcal{O}_M} \mathcal{E} \qquad \text{with}$$
$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s \qquad \text{for } f \in \Gamma(U, \mathcal{O}_M),\ s \in \Gamma(U, \mathcal{E}),\ U \subset M \text{ open}$$

2.1.2. *Examples of covariant derivations.*

Example 2.1. The differential $d : \mathcal{O}_M \longrightarrow \Omega^1_M$ is the canonical covariant derivation for $\mathcal{O}_M$.

Analogously for a vector space $V/\mathbb{R}$ is $d : \mathcal{V}_M \longrightarrow \Omega^1_M \otimes \mathcal{V}_M$ defined. Here $\mathcal{V}_M$ is the sheaf of section of the globally split vector bundle $M \times V \to M$. It is a free $\mathcal{O}_M$–module of rank $= \dim V = n$ and for a choice of a basis $\mathcal{V}_M \simeq \mathcal{O}^n_M$. The differential is obviously independent of the chosen basis.

2.1.3. *Extension of covariant derivation to the differential complex.* ∇ is defined on the complex $\Omega^\bullet_M \otimes_{\mathcal{O}_M} \mathcal{E}$ by

$$\nabla : \Omega^q_M \otimes \mathcal{E} \longrightarrow \Omega^{q+1}_M \otimes \mathcal{E}$$
$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^q \alpha \wedge \nabla s$$

(this ∇ is also called *exterior* covariant derivation [3, XVII.19]).

This definition is the obvious generalisation of the differential (exterior derivation) for forms with values in a vector bundle.

2.1.4. *Formula for the exterior covariant derivation on differential forms.* For a section in $\mathcal{E}$, $s \in \Gamma(U, \mathcal{E})$, the derivation $\nabla s \in \Gamma(U, \Omega^1_M \otimes \mathcal{E})$ is a 1–form with values in E, the vector bundle belonging to $\mathcal{E}$ (see 0.1.6). The evaluation of this form at a tangent vector $x \in T_m M$ is denoted by $\nabla_x s$, it is an element of the fiber E_m. For a tangent vector field $X \in \Gamma(M, TM)$ the analogous notation $\nabla_X s \in \Gamma(U, E)$ describes the section defined by $(\nabla_X s)_m = \nabla_{X_m} s$.

Similar to the formula for the ordinary differential d on forms we have: for ω a k–form with values in E, $X_0, \dots, X_k \in \Gamma(M, TM)$

$$\nabla\omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i \nabla_{X_i}(\omega(X_0, \dots, \widehat{X_i}, \dots, X_k)) + \\ + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k) \tag{11}$$

This can be calculated directly by making use of the corresponding formula for d, which we will not reproduce here (compare [9, §1.7], or [3]).

2.1.5. *Existence of covariant derivations.*

(1) There exists always a covariant derivation. Locally you can take the differential as in example 2.1. So, let ∇_α be defined on U_α, $M = \bigcup_\alpha U_\alpha$, and let $(\varphi_\alpha)_\alpha$ be an associated partition of one, and define $\nabla = \sum_\alpha \varphi_\alpha \cdot \nabla_\alpha$. This is $\mathbb{R}$–linear and satisfies the relation in the definition because $\sum_\alpha \varphi_\alpha = 1$, therefore $\sum_\alpha d\varphi_\alpha = 0$.

(2) The difference of two covariant derivations ∇, ∇' is obviously even $\mathcal{O}_M$–linear (the term $df \otimes s$ cancels). By the Lemma 0.1.3 we can interpret

$$\Phi = \nabla' - \nabla : \mathcal{E} \longrightarrow \Omega^1_M \otimes \mathcal{E}$$

as a section in $\Omega^1_M \otimes \mathcal{E}nd(\mathcal{E})$, that is a 1–form with values in the endomorphisms bundle $END(E)$.

2.1.6. *Curvature Form.*

Proposition 2.1.

$$\nabla\nabla : \mathcal{E} \longrightarrow \Omega^2_M \otimes \mathcal{E} \quad \textit{is } \mathcal{O}_M\textit{–linear !}$$

Proof. $\nabla\nabla(f \cdot s) = \nabla(df \otimes s) + \nabla(f \cdot \nabla s) = ddf \otimes s - df \wedge \nabla s + df \wedge \nabla s + f \cdot \nabla\nabla s = f \cdot \nabla\nabla s$, as $dd = 0$ □

Again by the Lemma 0.1.3 we can interpret $\nabla\nabla$ as a 2–form with values in the endomorphism bundle $END(E)$:

Definition 2.2. The 2–form defined by $\nabla\nabla$

$$\Omega \in \Gamma(M, \Omega^2_M \otimes END(E))$$

is called *curvature form* of ∇.

If the curvature vanishes, $\Omega = 0$, i.e. $\nabla\nabla = 0$, then ∇ is called *flat* (or *integrable*, see 2.2.3 for the geometric meaning). For example, the differential d is flat.

Let $X, Y \in \Gamma(M, TM)$ be tangential vector fields, then $\Omega(X, Y)$ is a section in $END(E)$. Let s be a section of the vector bundle E, then we have

$$\Omega(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

As $\Omega(X,Y)s = (\nabla\nabla s)(X,Y) = \nabla_X(\nabla s(Y)) - \nabla_Y(\nabla s(X)) - \nabla s([X,Y])$ by the formula in 2.1.4 for $k = 1$, $\omega = \nabla s$, qed.

For a covariant derivation in the tangent bundle this coincides with the usual definition of the curvature tensor (see [6, 2.2, (4)], or [7, I §8]).

2.1.7. *Induced covariant derivation.* Let E_1, E_2 be two vector bundles with covariant derivations ∇_1, ∇_2. Then there will be induced in a canonical way covariant derivations in the bundles $E_1 \oplus E_2$, $E_1 \otimes E_2$, $HOM(E_1, E_2)$ (see [9, 1.2]).

We will only need this for $\mathcal{H}om$:

$$\nabla : \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2) \longrightarrow \Omega^1_M \otimes \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)$$

For sections f in $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)$, s in $\mathcal{E}_1$ this is defined to be

$$(\nabla f)(s) := \nabla_2(f(s)) - f(\nabla_1 s)$$

In particular this gives us for the endomorphism bundle of a (E, ∇)

$$\begin{aligned} (\nabla_X f)s &= \nabla_X(f(s)) - f(\nabla_X s) && \text{or short} \\ \nabla_X f &= \nabla_X \circ f - f \circ \nabla_X && \text{where } X \in \Gamma(M, TM) \end{aligned}$$

2.1.8. *Variation of covariant derivations.* With [,] we denote the usual Lie *bracket* in $\mathcal{E}nd(\mathcal{E})$: $[f,g] = f \circ g - g \circ f$. For a 1–form ϕ with values in $END(E)$ we define a 2–form $[\phi, \phi]$ with value in $END(E)$ by $[\phi,\phi](X,Y) := [\phi(X), \phi(Y)]$ (this is a form, as the right hand side is alternating in (X,Y)).

When two covariant derivations ∇, ∇' are given on E with curvature forms Ω, Ω' and if we denote with ∇ also the induced covariant derivation on $END(E)$, then we have

$$(*) \qquad \Omega' = \Omega + \nabla\phi + [\phi,\phi]\ , \quad \text{where } \phi = \nabla' - \nabla$$

Proof. (Koszul):

$$\begin{aligned} \Omega'(X,Y) &= \nabla'_X \nabla'_Y - \nabla'_Y \nabla'_X - \nabla'_{[X,Y]} = \\ &= (\nabla_X + \phi(X))(\nabla_Y + \phi(Y)) - (\nabla_Y + \phi(Y))(\nabla_X + \phi(X)) - \nabla_{[X,Y]} - \phi([X,Y]) = \\ &= \Omega(X,Y) + \nabla_X(\phi(Y)) - \nabla_Y(\phi(X)) - \phi([X,Y]) + [\phi(X), \phi(Y)] = \\ &= \Omega(X,Y) + \nabla\phi(X,Y) + [\phi,\phi](X,Y) \end{aligned}$$

by (2.1.6), (2.1.7) and (2.1.4) □

2.2. **Connections on principal and fiber bundles.**

2.2.1. *Pulled back forms on vector bundles.* We come back to section 0.1.8. There we defined the pull back of forms with values in vector bundles.

$$\pi : P \longrightarrow M \qquad \text{smooth,} \qquad E/M \quad \text{vector bundle}$$

$$(12) \qquad \pi^* : \mathrm{Hom}_M(\Lambda^k TM, E) \longrightarrow \mathrm{Hom}_P(\Lambda^k TP, \pi^* E)$$

When π is a surjective submersion, then (12) is injective, as you recognize from the explicit formula in section 0.1.8. A pulled back form yields 0, when applied to vertical vectors (it suffices that one of them is vertical). To determine the exact image of (12), you need to compare vectors at different base points in the same fiber of π. In general it is not clear, when a form on P with values in $\pi^* E$ is a pulled back form (we will then say, that it "comes from M"). The situation is satisfactory, when $\pi : P \longrightarrow M$ is a *principal* bundle.

2.2.2. *Principal bundles.* Let $\pi : P \longrightarrow M$ be a principal bundle with structure group G. The LIE algebra of the LIE group G will be denoted $\mathfrak{g}$. G will be operating on the *right* on P. The group operation $P \times G \longrightarrow P$ yields by second partial derivation (in the unit element $e \in G$)

$$\begin{aligned} P \times \mathfrak{g} &\longrightarrow TP \\ (p, a) &\longmapsto pa \end{aligned} \tag{13}$$

and by first partial derivation

$$\begin{aligned} TP \times G &\longrightarrow TP \\ (x, g) &\longmapsto xg \end{aligned} \tag{14}$$

The chain rule shows, that (14) is an operation of G on TP. The orbit of a $p \in P$ under G is equal to the fiber in which p lies: $pG = P_m = \pi^{-1}(m)$ $(m = \pi(p))$, therefore the image of $\{p\} \times \mathfrak{g}$ in (13) is $T_p(P_m) = V_p$, the space of vertical vectors at p, and the map (13) becomes

$$P \times \mathfrak{g} \xrightarrow{\sim} V \subset P$$

The exact sequence (1) in section 0.3.2 now reads

$$0 \to P \times \mathfrak{g} \longrightarrow TP \longrightarrow \pi^*TM \to 0 \tag{15}$$

G operates on π^*E by the operation on P by moving the fibers of π^*E along the fibers of P: the fiber of π^*E at $p \in P_m$ is $(\pi^*E)_p = \{p\} \times E_m$ and $g \in G$ has the effect $(\pi^*E)_p \cdot g = (\pi^*E)_{pg}$.

After these prolegomena we prove the

Lemma 2.2. *A form $\eta \in \mathrm{Hom}_P(\bigwedge^k TP, \pi^*E)$ comes from M if and only if for all $x_1, \ldots, x_k \in T_pP$*

(1) $\eta_p(x_1, \ldots, x_k) = 0$ *if one* $x_i \in V_p$
(2) $\eta_{pg}(x_1 g, \ldots, x_k g) = \eta_p(x_1, \ldots, x_k) \cdot g \quad \forall\, g \in G$

Proof. As we have $\pi(pg) = \pi(p)$ it follows by taking the derivative that $\pi_*(xg) = \pi_*(x)$, and therefore a pulled back form satisfies the conditions (1), (2) (see the formula in section 0.1.8).

Let conversely η with conditions (1), (2) be given. Define for $m \in M$, $t_1, \ldots, t_k \in T_mM$ the form $\omega \in \mathrm{Hom}_M(\bigwedge^k TM, E)$ as follows: choose a $p \in P_m$ and $x_1, \ldots, x_k \in T_pP$ with $\pi_*(x_i) = t_i$

$$\omega_m(t_1, \ldots, t_k) := pr(\eta_p(x_1, \ldots, x_k)) \in E_m \tag{$*$}$$

where $pr : \pi^*E \longrightarrow E$ is the canonical map (this definition is necessary, cf. 0.1.8). Because of condition (1) the right hand side is independent from the choice of the x_i; from condition (2) follows the independence from the chosen p above m. □

2.2.3. *Geometrical meaning of a connection.* Before I give the formal definition, I want to explain what a connection on P is in intuitive geometric terms.

To comfortably study the infinitesimal properties of $P \longrightarrow M$, you want to decompose a vector at P into a vertical component (which is to be discarded under π_*) and into a "horizontal" component, i.e. you are looking for a splitting $T_pP = V_p \oplus H_p$. Of course, you do get this by looking at a submanifold thru p in P, which is transversal to the fiber and take its tangent space. But it is far from clear, that the gained H_p spaces can be glued together into a bundle $H \subset TP$. There even is a global obstruction. In the FROBENIUS theory ([10, chap. 6]) you call a vector bundle $H \subset TP$

with this property, to be locally a tangent bundle to a submanifold, *integrable*. One criterion for it is: with $X, Y \in \Gamma(P, H)$ you must have $[X, Y] \in \Gamma(P, H)$.

A *connection* on P is defined as a global infinitesimal split of P/M, which is G–invariant, i.e.

$$TP = V \oplus H\ , \quad \forall\, g \in G \quad H_p \cdot g = H_{pg}$$

(we have anyhow $V_p \cdot g = V_{pg}$).

We will see, that this always exists, but H is in general not integrable. One can associate a differential operator ∇ to a connection, which is flat exactly when H is integrable (the curvature measures the deviation). Equivalent to this notion is obviously that the sequence (15) in 2.2.2 is split-exact, and this G-invariantly. The retraction $TP \longrightarrow P \times \mathfrak{g}$ can thus be interpreted as 1–form with values in $\mathfrak{g}$.

2.2.4. *Definition of a connection.*

Definition 2.3. A 1–form on P with values in $\mathfrak{g}$

$$\omega : TP \longrightarrow \mathfrak{g}$$

is called a *connection form*, if it satisfies these condition

(1) $\forall\, p \in P, a \in \mathfrak{g} \quad \omega_p(pa) = a$
(2) $\forall\, x \in T_pP, g \in G \quad \omega_{pg}(xg) = g^{-1} \cdot \omega_p(x) \cdot g$

The kernel of $\omega_p : T_pP \longrightarrow \mathfrak{g}$ is denoted by $H_p = \operatorname{Ker} \omega_p$, $H = \bigcup H_p \subset TP$ is the bundle of *horizontal* vectors.

The following is clear: $TP = V \oplus H, \forall\, g \in G \quad H_{pg} = H_p \cdot g$.
$\pi_* : H_p \xrightarrow{\sim} T_mM$, where $m = \pi(p)$.

2.2.5. *Covariant derivative of a connection form.* The differential d is a covariant derivation for the vector space $\mathfrak{g}$ (example 2.1). Every other is different from d by a 1–form with values in $\operatorname{End}(\mathfrak{g})$, by (2) in section 2.1.5. Let

$$\operatorname{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) = \operatorname{End}(\mathfrak{g})$$

be the adjoint representation of the Lie algebra $\mathfrak{g}$. By $\operatorname{ad} \circ \omega$ there is given a 1–form on P with values in $\operatorname{End}(\mathfrak{g})$.

Definition 2.4. The *covariant derivation* (for $\mathfrak{g}$) of the the connection form ω is defined as

$$\nabla = d + \operatorname{ad} \circ \omega$$

What is its curvature form ?

The curvature of d is 0. By ($*$) in section 2.1.8 the curvature Ω' of ∇ is

$$\Omega' = d(\operatorname{ad} \circ \omega) + [\operatorname{ad} \omega, \operatorname{ad} \omega] = \operatorname{ad}(d\omega) + \operatorname{ad}([\omega, \omega])$$

(as "ad" is a homomorphism of Lie algebras by the Jacobi identity). Therefore $\Omega' = \operatorname{ad}(d\omega + [\omega, \omega])$. Because we do not want to loose information by "ad", we define

Definition 2.5. The *curvature form* of the connection form ω is

$$\Omega = d\omega + [\omega, \omega]$$

Ω is a 2–form on P with values in $\mathfrak{g}$.

2.2.6. *Existence of connections on principal bundles.* By second partial derivative of the multiplication $G \times G \longrightarrow G$ we obtain $G \times \mathfrak{g} \longrightarrow TG$. For $g \in G$ we define conversely $T_g G \longrightarrow G \times \mathfrak{g}$ by $x \longmapsto (g, g^{-1}x)$. We will thus identify $TG = G \times \mathfrak{g}$.

The projection $\alpha : TG \longrightarrow \mathfrak{g}$ is a 1–form on G with values in $\mathfrak{g}$.

Example: $G = \mathbb{S}^1 \subset \mathbb{C}$, $\mathfrak{g} = T_1\mathbb{S}^1 = i\mathbb{R} \subset \mathbb{C}$; with $z = x + iy$ we have — compare 1.2.5 — by definition

$$\alpha = z^{-1}dz = \frac{\bar{z} \cdot dz}{\bar{z} \cdot z} = \frac{xdx + ydy}{x^2 + y^2} + i\frac{xdy - ydx}{x^2 + y^2} = d\log r + i(xdy - ydx) = i\sigma$$

Local situation: $P = M \times G$, let $\pi_2 : P \longrightarrow G$ be the projection. Assertion: $\omega = \pi_2^*(\alpha)$ is a connection form on P. Proof: We have $\omega : TP = TM \times G \times \mathfrak{g} \xrightarrow{\pi_{2*}} G \times \mathfrak{g} \xrightarrow{\alpha} \mathfrak{g}$

G operates on TP by: $t \in TP$, $h \in G$, $a \in \mathfrak{g}$ $(t, h, a) \cdot g = (t, hg, g^{-1}ag)$ and the condition (1),(2) in section 2.2.4 are clear, qed.

Global situation: Let ω_α be locally defined connection forms (in $U_\alpha \subset P$, for example). Choose an associated partition $(\varphi_\alpha)_\alpha$ of one on P. With $\omega = \sum_\alpha \varphi_\alpha \cdot \omega_\alpha$ the conditions of section 2.2.4 are again fulfilled.

2.2.7. *Theorem.*

Theorem 2.3. *Let Ω be the curvature form of a connection form ω on the principal bundle P. Then we have*

(1) $\forall\, p \in P,\ x \in V_p,\ y \in T_pP \quad \Omega_p(x, y) = 0$
(2) $\forall\, p \in P,\ x, y \in T_pP,\ g \in G \quad \Omega_{pg}(xg, yg) = g^{-1} \cdot \Omega_p(x, y) \cdot g$

Proof. The proof will be postponed to section 2.2.8. It will be accomplished by continuation of tangential vectors to properly chosen vector fields. To this end we'll need first two lemmas.

Lemma 2.4. *Let $y \in H_p$ be a horizontal vector. Then there exists a horizontal vector field $Y \in \Gamma(P, H)$ with*

$$Y_p = y \quad \textit{and} \quad Y_{qg} = Y_q \cdot g \quad \textit{for all} \quad q \in P, g \in G$$

Proof. The idea is that such a vector field exists locally over a section of M in P, and this is translated by the group operation along the fibers of P over M.

Let $\pi(p) = m \in M$, $U \subset M$ a suitable open neighbourhood of m, such that there exists a section $s : U \longrightarrow P$ with $s(m) = p$. Let $t = \pi_*(y) \in T_mM$ and take a vector field $T : U \longrightarrow TM$ of M with $T_m = t$ and such that the support is contained in U.

For all $u \in U$ we have $\pi_* : H_{s(u)} \xrightarrow{\sim} T_uM$ and the vector field T can be transported back along the section s to a smooth horizontal vector field $Y : s(U) \longrightarrow H$ with $Y_p = y$.

Continue Y on the fibers over U by $Y_{qg} := Y_q \cdot g$, $q \in s(U)$, and outside of $\pi^{-1}(U)$ by 0. As G operates smoothly, the field Y is smooth as well. □

Lemma 2.5. *Let X be a vertical vector field $X \in \Gamma(P, V)$ of the following kind: there exists a $v \in \mathfrak{g}$ with $X_q = qv$ for all $q \in P$.*

Let Y be a G–invariant vector field: $Y \in \Gamma(P, TP)^G$, that is $Y_{qg} = Y_q \cdot g$ for all $q \in P, g \in G$.

Then we have $[X, Y] = 0$.

Proof. The LIE bracket is calculated by [10, chap. 5 §1]. We need a local flow for the vector field, which we actually can find globally. Let $\exp : \mathfrak{g} \longrightarrow G$ be the exponential mapping for the LIE group (see [7, chap. II, §1, Cor. 1.5 f.]) and let

$$\Phi : \mathbb{R} \times P \longrightarrow P$$

$$\Phi(t, q) = q \cdot \exp(tv)$$

Assertion: Φ is an integral flow for the vector field X.

(1) Initial condition: $\Phi(0, q) = q \cdot \exp(0) = q$
(2) Integral condition: $\frac{\partial \Phi}{\partial t}(0, q) = qv = X_q$

Furthermore we have with $\Phi_t : P \to P \quad \Phi_t(q) = \Phi(t, q)$

$$((\Phi_{-t})_* \circ Y \circ \Phi_t)_q = Y_{\Phi(t,q)} \cdot \exp(-tv) = Y_{q \exp(tv)} \cdot \exp(tv)^{-1} = Y_q$$

by assumption on Y. Therefore $(\Phi_{-t})_* \circ Y \circ \Phi_t = Y$ is not dependent on t. By Lang loc.cit. we have

$$[X, Y] = \frac{d}{dt}|_{t=0}(\Phi_{-t})_* \circ Y \circ \Phi_t = 0$$

□

2.2.8. *Proof of Theorem 2.3.* The vanishing relation (1) $\Omega_p(x, y) = 0$

By linearity (in y) it suffices to prove the vanishing for the two cases of a vertical and a horizontal vector.

The vertical case: $y \in V_p$.

x and y can be written $x = pv, y = pw$ for certain $v, w \in \mathfrak{g}$ by the exact sequence (15) in section 2.2.2. Define $X, Y \in \Gamma(P, V)$ by

$$\forall\, q \in P \qquad X_q := qv, \quad Y_q := qw$$

We have to calculate $\Omega_p(x, y) = (\Omega(X, Y))_p$. Now we have

$$(*) \quad \Omega(X, Y) = d\omega(X, Y) + [\omega, \omega](X, Y) =$$
$$= X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) + [\omega(X), \omega(Y)]$$

But $\omega(X) : P \longrightarrow \mathfrak{g}$ is constant, as we have

$$(\omega(X))_q = \omega_q(X_q) = \omega_q(qv) = v$$

and analogously for $\omega(Y)$. The first two terms in $(*)$ vanish therefore, the last term is the constant function $[v, w]$. We again have to calculate $[X, Y]$: with the notations as in Lemma 2.5 we have

$$((\Phi_{-t})_* \circ Y \circ \Phi_t)_q = Y_{q \exp tv} \cdot \exp(-tv) = q \cdot \exp(tv) \cdot w \cdot \exp(tv)^{-1}$$

Now $\exp(tv) \cdot w \cdot \exp(tv)^{-1} = \mathrm{Ad}\,(\exp tv)(w)$ where

$$\mathrm{Ad}\; : G \longrightarrow GL(\mathfrak{g})$$

is the adjoint representation of the LIE group. By [7, chap. II, §5, (4)] we have

$$\mathrm{Ad}\,(\exp tv)(w) = w + t \cdot [v, w] + O(t^2)$$

and therefore

$$[X, Y]_q = \frac{d}{dt}|_{t=0}((\Phi_{-t})_* \circ Y \circ \Phi_t)_q = q[v, w]$$

which implies

$$\omega_q([X, Y]_q) = \omega_q(q[v, w]) = [v, w]$$

and we are done for the vertical case.

The horizontal case: $y \in H_p$

Choose X as above, Y from Lemma 2.4. We have $\omega(Y) = 0$ and $(*)$ above shows with Lemma 2.5 that $\Omega_p(x, y) = 0$.

The operation of G relation (2) in Theorem 2.3.

According to what we have just proven it suffices to calculate this for horizontal vectors x, y. Choose vector fields X, Y as in Lemma 2.4. As for all $g \in G$ the morphisms $P \xrightarrow{\sim} P$, $q \mapsto q \cdot g$ are diffeomorphisms, we have by [7, chap. I, §3, prop. 3.3] the relation

$$[Xg, Yg] = [X, Y]g$$

where Xg is the field $(Xg)_q = X_{qg^{-1}}g$.

From $(*)$ we conclude $\Omega(X, Y) = -\omega([X, Y])$, as X, Y are horizontal fields. By construction we have $X_q g = X_{qg}$, i.e. $Xg = X$, as well $Yg = Y$, $[X, Y]$ fulfills the same relation $[X, Y]_{qg} = [Xg, Yg]_{qg} = ([X, Y]g)_{qg} = [X, Y]_q g$. Therefore

$$\begin{aligned}\Omega_{pg}(xg, yg) = (\Omega(X, Y))_{pg} = -\omega_{pg}([X, Y]_{pg}) = -\omega_{pg}([X, Y]_p g) = \\ = -g^{-1} \cdot \omega_p([X, Y]_p) \cdot g = g^{-1} \cdot \Omega_p(X_p, Y_p) \cdot g\end{aligned}$$

□

Note. For horizontal fields $X, Y \in \Gamma(P, H)$ we know that $\omega([X, Y]) = -\Omega(X, Y)$. The vertical component of the Lie bracket $[X, Y]$ – the deviation of being horizontal – is measured by the curvature (recall the discussion in section 2.2.3).

2.2.9. *Induced connection in fiber bundle.* Similar to the situation for principal bundles there are again two equivalent methods. We will start with the direct splitting into vertical and horizontal bundles.

Let $\pi : P \longrightarrow M$ be a principal bundle with connection, F a G–space on which G operates as usual on the left and let $E = (P \times F)/G$ the associated fiber bundle over M with typical fiber F. As everybody knows, the following diagram is cartesian

$$\begin{array}{ccc} P \times F & \xrightarrow{\pi_F} & E \\ \downarrow & & \downarrow \\ P & \xrightarrow{\pi} & M \end{array}$$

that is $P \times F \xrightarrow{\sim} \pi^* E = P \underset{M}{\times} E$.

We look at the infinitesimal situation. Let $p \in P$, $a \in F$, $e = (p, a)G \in E$ the image point and $m = \pi(p)$.

$$\begin{array}{ccc} T_pP \times T_aE & \longrightarrow & T_eE \\ \downarrow & & \downarrow \\ T_pP & \longrightarrow & T_mM \end{array}$$

The vertical space $V_eE \subset T_eE$ of E/M is $T_e(E_m)$, where E_m is the fiber of E over $m \in M$. As we have by the cartesian diagram

$$\{p\} \times F \xrightarrow{\sim} E_m$$

we also have by second partial derivative

$$\{p\} \times T_aF \xrightarrow{\sim} V_eE$$

$$(p, y) \longmapsto py$$

Define $H_eE \subset T_eE$ as the image of $H_p \times \{a\}$ under the mapping (first partial derivative)

$$T_pP \times F \longrightarrow TE$$
$$(x, a) \longmapsto xa$$

As the projection onto T_mM induces an isomorphism $H_p \times \{a\} \xrightarrow{\sim} T_mM$, we have $H_p \times \{a\} \xrightarrow{\sim} H_eE$. Explicitely

$$V_eE = \{py \mid y \in T_aF\}$$
$$H_eE = \{xa \mid x \in H_p\}$$

This definition is independent of the pair (p, a) over e: another pair lies in the same orbit under G, there is a $g \in G$ such that this second pair is $(p, a) \cdot g = (pg, g^{-1}a)$. As $H_{pg} = H_pg$ and $xg \cdot g^{-1}a = xa$, we see this independence.

Thus we defined on the fiber bundle a tangential splitting

$$TE = VE \oplus HE$$

An equivalent description is given by *vertical projection* $K : TE \longrightarrow VE$.

By derivation of the operation $G \times F \longrightarrow F$ we get

$$\mathfrak{g} \times TF \longrightarrow TF$$
$$(v, y) \longmapsto vy = y + va$$

(where $y \in T_aF$) by the formula for partial differentiation (see [3, 16.6.6]).

$$\begin{array}{ccc} TP \times TF & \longrightarrow & TE \\ \downarrow{\scriptstyle \omega\times id} & & \\ P \times \mathfrak{g} \times TF & & \downarrow{\scriptstyle K} \\ \downarrow & & \\ P \times TF & \longrightarrow & VE \end{array}$$

For $e \in E$, $t \in T_eE$ choose a $(p, a) \in P \times F$ over e, $x \in T_pP$, $y \in T_aF$ such that $t = py + xa$. Then K is given by

$$K_e : T_eE \longrightarrow V_eE$$
$$K(t) = py + p\omega_p(x)a$$

2.2.10. *The case of vector bundles.* When F is in particular a vector space, on which G operates *linearly*, i.e. E is a *vector bundle*, then the vertical space becomes $V_eE = T_e(E_m) = \{e\} \times E_m$, and $VE = E \underset{M}{\times} E$ (see section 0.3.3).

In this case the map $pr \circ K : TE \longrightarrow VE \longrightarrow E$ is also called *connection mapping*. We will again designate this with K.

For $\varepsilon \in \mathbb{R}$ we denote the homothety with ε by $\varepsilon : E \longrightarrow E$, and $\varepsilon_* : TE \longrightarrow TE$ its derivative.

Lemma 2.6. *$K \circ \varepsilon_* = \varepsilon \circ K$*

$$\begin{array}{ccc} TE & \xrightarrow{\varepsilon_*} & TE \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle K} \\ E & \xrightarrow{\varepsilon} & E \end{array}$$

Proof. Here we have $TF = F \times F$ and ε_* is on $F \times F$ just the homothety by ε on both factors. Write a $t \in TE$ as $t = xa + py$. From the description in section 2.2.9 follows $K(\varepsilon_*t) = K(x \cdot \varepsilon a + p \cdot \varepsilon y) = K(\varepsilon t) = \varepsilon K(t)$. □

2.2.11. *Relation of connection on principal and vector bundle.* The connection mapping $K : TE \longrightarrow VE$ is a 1–form on E with values in VE. What is the pulled back form $\pi_F^*(K)$? It is a 1–form on $P \times F$ with values in $\pi_F^* VE = P \times F \times F$:

$$TP \times F \times F \longrightarrow P \times F \times F$$

$$(x, a, y) \longmapsto (p, a, y + \omega_p(x)a)$$

The pulled back connection mapping thus yields the exact knowledge of the connection form, if only $G \hookrightarrow GL(F)$ is injective (faithful representation), and therefore also $\mathfrak{g} \hookrightarrow \mathfrak{gl}(F)$.

3. A representative of the Thom class

The theory presented in section 2 finds two applications here:

In section 3.1 we will construct forms on the total space of a fiber bundle by extending forms on the typical fiber to all fibers by means of a connection.

In section 3.4 we will construct forms of higher degree starting with the scalar valued connection and curvature forms. These will eventually enable us to solve the differential equations of theorem 1.2.

3.1. **Fiber forms on fiber bundles.**

3.1.1. *Extending fiber forms with a connection.*

Definition 3.1 (Notations as in section 2.2.9)**.** To each G–invariant form on the typical fiber F we can associate a relative form

$$\Omega^k(F)^G \longrightarrow \Omega^k_{E/M}(E) \tag{16}$$

Note. G operates on $\Omega^k(F)$ (on the right) by pull back:
$\forall\, g \in G \quad g : F \longrightarrow F \qquad \alpha.g := g^*(\alpha)$.

The searched for form ω will first be defined on the fiber E_m by $\omega|E_m := (p^{-1})^*(\alpha)$ for a $p \in P_m$, $p : F \xrightarrow{\sim} E_m$. Because of the G–invariance of α this does not depend on the choice of the $p \in P_m$: $((pg)^{-1})^*(\alpha) = (p^{-1})^*(g^{-1})^*(\alpha) = (p^{-1})^*(\alpha)$.

Thus we have defined a section $\omega : E \longrightarrow \bigwedge^k V^*E$, cf. section 0.3.2. To see that it is smooth, let $s : U \to P$ be a local section and $E|U \simeq U \times F$ the induced trivialisation: $\omega|_{E|U} : E|U \simeq U \times F \xrightarrow{1\times\alpha} U \times \bigwedge^k T^*F \simeq \bigwedge^k V^*E|_U$, qed.

Now let a connection on E be given, e.g. by a vertical projection $K : TE \longrightarrow VE$. This gives rise to a bundle morphism $K^* : \bigwedge^k V^*E \longrightarrow \bigwedge^k T^*E$ and composed with (16) a map

$$\Omega^k(F)^G \longrightarrow \Omega^k(E)$$

$$\alpha \longrightarrow K^*(\omega)$$

where ω is the relative form we constructed in (16).

Definition 3.2. Let a canonical G–invariant volume form on F be given. The corresponding form on E is called *fiber volume form*.

3.1.2. *Fiber volume forms on a Riemannian vector bundle.* Now let (E, g) be an oriented Riemannian vector bundle. The structure group is $G = SO(n)$ (n = fiber dimension), the typical fiber is $F = \mathbb{R}^n$, the principal bundle has the fibers $P_m = \{p : \mathbb{R}^n \xrightarrow{\sim} E_m \mid p \text{ is oriented isometry}\}$.

The form $dx^1 \wedge \dots \wedge dx^n$ is $SO(n)$–invariant, the corresponding fiber volume form will be denoted by v.

An equivalent definition to ours in section 3.1.1 is given by Ferus [4, 1.9]:

Let $\mu : M \longrightarrow \bigwedge^n E^*$ be the section, which maps each $m \in M$ to the Riemannian volume form $\mu_m \in \bigwedge^n E^*_m$ in each fiber. Then the following diagram is commutative

$$\begin{array}{ccc} \Lambda^n T^*E & \xleftarrow{K^*} & \Lambda^n E^* \\ \uparrow v & & \uparrow \mu \\ E & \longrightarrow & M \end{array}$$

This is evident (compare also the local expression in section 3.1.5).

3.1.3. *Fiber volume forms on a Riemannian sphere bundle.* We generalize also the form σ considered in section 1.2.5.
The form $\sum_i (-1)^{i+1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$ in $\mathbb{R}^n$ is $SO(n)$–invariant (you can either see this by calculation – or by the geometric argument, that this form $/r$ is the Riemannian fiber volume form on each sphere around 0 (section 1.2.5). The corresponding form on E will now be denoted by σ. Obviously is σ/r the fiber volume form of each sphere bundle $S(\delta) := \{e \in E \mid \|e\| = \delta\}$ around the zero section (with the notations of section 1.2.3, $\delta > 0$).

3.1.4. *Occurance of curvature.*

Note. All the relations proven in section 1.2.5 for the Euclidean case become wrong for general vector bundles:

- $d\sigma = n.v+$ terms containing the curvature form of the covariant derivation of E
- $d\frac{\sigma}{r^n} = \frac{1}{r^n}\cdot$ curvature terms
- $\varphi^*(\sigma) = \sigma/r^n +$ terms in dr and curvature

3.1.5. *Local representations of fibre volume forms by connection forms.* We need the local expressions for the forms v and σ.

Let $s : U \longrightarrow P$ be a local section in the principal bundle, it defines n oriented, orthonormal sections in the vector bundle

$$s_1, \dots, s_n : U \longrightarrow E \quad \text{by}$$
$$s_i(m) := s(m)e_i \quad e_i \in \mathbb{R}^n \text{ canonical basis}$$

(it is $s(m) \in P_m$, i.e. $s(m) : \mathbb{R}^n \xrightarrow{\sim} E_m$ is positively oriented isometry)

Let $y^1, \dots, y^n : U \longrightarrow E^*$ be the dual sections, taken as fiber coordinates (section 0.1.2): $y^i : E|U \longrightarrow \mathbb{R}$. These relations hold: $e = \sum_i y^i(e) \cdot s_i(m) \quad \forall e \in E_m$.

From the local description (section 3.1.1)

$$\begin{array}{ccc} U \times \mathbb{R}^n & \longrightarrow & U \times \Lambda^n T^*\mathbb{R}^n \\ \downarrow \simeq & & \downarrow \simeq \\ E|U & \longrightarrow & E|U \times_U \Lambda^n E^* \end{array}$$

follows, that to the form $dx^1 \wedge \cdots \wedge dx^n$ corresponds the relative form $y^1 \wedge \cdots \wedge y^n$, and therefore we get

$$v|_{E|U} = \kappa^1 \wedge \cdots \wedge \kappa^n \tag{17}$$

where $\kappa^i := y^i \circ K : TE \to E \to \mathbb{R}$.

Analogously we get for σ:

$$\sigma|_{E|U} = \sum_i (-1)^{i+1} y^i \cdot \kappa^1 \wedge \cdots \wedge \widehat{\kappa^i} \wedge \cdots \wedge \kappa^n \tag{18}$$

The κ^i are locally on $E|U$ defined 1–forms. We will call them *local connection forms.*

3.1.6. *Local relations of the connections forms.* Let us indicate how the calculations for the assertions in section 3.1.4 can be carried thru.

As the LIE algebra is $\mathfrak{so}(n) \subset \mathfrak{gl}(n) = \mathrm{End}(\mathbb{R}^n)$, similar to section 2.2.5, by $\nabla := d + \omega$ a covariant derivation is defined for $\pi^* E = P \times \mathbb{R}^n$. The pull back gives

$$\Omega^k_M \otimes E \longrightarrow \pi_* \Omega^k_P \otimes \mathbb{R}^n$$

It is easy to see that the differential operator ∇ on $\Omega^k_P \otimes \mathbb{R}^n$ can be defined on $\Omega^k_M \otimes E$. This coincides with the usual definition $\nabla s = K \circ s_*$ (see [6, 2.5], [4, Satz 1.3, (i)]). With the notations from 3.1.5 it is easy to see that

$$\kappa^i = dy^i + \sum_j \omega_{ij} \cdot y^j$$

where (ω_{ij}) is the matrix of the connection form. The curvature terms show up, when we differentiate the κ^i (this will become more explicit in section 3.4.2) – as it should be, as we differentiate covariantly twice and $\nabla\nabla$ is by definition the curvature.

3.1.7. *Relation with the approach by* FERUS. FERUS gives the following definition for σ (which he denotes ω^1, see [4, 1.9]): Let

$$A : E \xrightarrow{\Delta} E \underset{M}{\times} E \xrightarrow{\sim} VE \hookrightarrow TE$$

be the *radial vector field*. Then we have $\sigma = v \llcorner A$ (evaluate the vector in the form).

As obviously $K \circ A = id$, we get locally:

$$v \llcorner A = (\kappa^1 \wedge \cdots \wedge \kappa^n) \llcorner A = \sum_i (-1)^{i+1} \kappa^i \circ A \cdot \kappa^1 \wedge \cdots \wedge \widehat{\kappa^i} \wedge \cdots \wedge \kappa^n =$$

$$= \sum_i (-1)^{i+1} y^i \cdot \kappa^1 \wedge \cdots \wedge \widehat{\kappa^i} \wedge \cdots \wedge \kappa^n = \sigma \quad \text{qed.}$$

(cf. [3, 16.18.4.7])

3.2. **Forms with poles in the zero section.** Let (E, g) be a Riemannian vector bundle on a Riemannian manifold M with a connection $K : TE \longrightarrow E$. Then E itself is a Riemannian manifold in a canonical way by the following definition of a metric in the tangential bundle:

$$TE = HE \oplus VE$$

is defined as *orthogonal* splitting and

$$HE \simeq E \underset{M}{\times} TM$$

receives the metric from M

$$VE \simeq E \underset{M}{\times} E$$

receives the fiber metric g of E.

In particular, the ball and sphere bundle are Riemannian manifolds. In this way all tensor bundles on E, B, ... obtain a fiber metric. The length of a co-vector, tensors, is denoted by $\| \quad \|$.

3.2.1. *Poles of order k in the zero section.*

Definition 3.3. To each differential form on (say) B is associated a *norm* function $\|\omega\| : B \longrightarrow \mathbb{R}$ by setting $\|\omega\|_e := \|\omega_e\|$.

By definition of the norm in $\bigwedge^* TB$ we have

$$\|\omega\|_e = \sup_{\|x_i\|=1} |\omega_e(x_1, \ldots, x_k)| \tag{19}$$

for a k–form ω, $x_1, \ldots, x_k \in T_eB$ unit vectors.

With the notations of section 1.2.3 we define

Definition 3.4. A differential form ω on B_0 has a *pole of order k in the zero section*, if we have

$$\|\omega\| = O(r^{-k})$$

3.2.2. *The derivative of the polar coordinate.* The only poles that we will encounter in this paper are produced by pull back of forms on S via $\varphi : B_0 \longrightarrow S$. We therefore need its derivative.

Lemma 3.1. *The tangential map of $\varphi : B_0 \longrightarrow S$ is*

$$T_e\varphi(x) = \varepsilon_*^{-1}(x - dr(x) \cdot \frac{1}{\varepsilon} A(e))$$

where $x \in T_eB_0$, $\quad \varepsilon = \|e\| = r(e)$, $\quad A$ is the radial vector field.

The geometric meaning of the second term is clear: $\varepsilon^{-1} \cdot A(e)$ is the unit normal vector at point $e \in B_0$ (as $K(\varepsilon^{-1} \cdot A(e)) = \varepsilon^{-1} \cdot K \circ A(e) = \frac{1}{\varepsilon} e$), and $dr(x)$ is the length of the normal component (normal to $S \subset B$).

Proof. Let $v : E \underset{M}{\times} E \xrightarrow{\sim} VE$ be the canonical identification of the vertical bundle on E with the pulled back vector bundle of E to E (and don't take it for the fiber volume form v now). At a point $e \in E_m$ the map $\{e\} \times E_m \xrightarrow{v} V_eE = T_eE_m$ is the derivative of the fiber inclusion $j_m : E_m \hookrightarrow E$.

Now φ is just the composition of $(\frac{1}{r}, id) : E_0 \longrightarrow \mathbb{R} \times E$ and the scalar multiplication $\mu : \mathbb{R} \times E \longrightarrow E$. By the formula of partial differentiation we have

$$T_{(a,e)}\mu(b, x) = T_a\mu(-, e)(b) + T_e\mu(a, -)(x) \tag{$*$}$$

As $\mu(-, e) : \mathbb{R} \to E$ factors thru the fiber E_m, in which e lies, $\mathbb{R} \to E_m \to E$ is by definition of v:

$$T_a\mu(-, e)(b) = v(ae, be) \qquad a, b \in \mathbb{R}, \ e \in E$$

v is as derivative linear in the second argument. Furthermore, the diagram

$$\begin{array}{ccc} E_m & \hookrightarrow & E \\ \downarrow a & & \downarrow a \\ E_m & \hookrightarrow & E \end{array}$$

with homotheties by $a \in \mathbb{R}$ gives by differentiation

$$\begin{array}{ccc} E_m \times E_m & \xrightarrow{v} & E \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a_*} \\ E_m \times E_m & \xrightarrow{v} & E \end{array}$$

that is: $v(ae, ae') = a_* v(e, e')$, for $e, e' \in E_m$.

The second term in $(*)$ is $a_*(x)$. Then we have (for $a \neq 0$):

$$T_{(a,e)}\mu(b, x) = v(ae, be) + a_*(x) = \frac{b}{a} \cdot v(ae, ae) + a_*(x) = a_*(x + \frac{b}{a} \cdot A(e))$$

as $v(e, e) = A(e)$.

With $d\frac{1}{r} = -\frac{1}{r^2} dr$ the chain rule gives

$$T_e\varphi(x) = T_{(\varepsilon^{-1},e)}\mu \circ (d\frac{1}{r}, id)(x) = \varepsilon_*^{-1}(x - \frac{1/\varepsilon^2 \cdot dr(x)}{1/\varepsilon} \cdot A(e))$$

as asserted. □

3.2.3. *Tangential and normal vectors to the sphere bundle.* The following remark is clear: Let $e \in E_0$, $x \in T_eE$, $\varepsilon = r(e)$

- $dr(x) = 0 \Leftrightarrow x$ is tangential to $S(\varepsilon)$
- $T_e\varphi(x) = 0 \Leftrightarrow x$ is normal to $S(\varepsilon)$

The normal bundle of S in B, $N \longrightarrow S$, is given by $N_e = \mathbb{R} \cdot A(e)$ and $A|S : S \longrightarrow N$ is a global exterior unit normal field

$$\begin{array}{ccc} N \oplus TS & \hookrightarrow & TB \\ \downarrow & & \downarrow \\ S & \hookrightarrow & B \end{array}$$

3.2.4. *Poles of forms pulled back from the sphere.* We will now discuss the poles of forms $\varphi^*(\omega)$, where ω is a form on S. The order of the pole in the zero section is the number of vertical directions, in which ω is 'measuring something'. An ℓ–form ω is said to *measure k vertical directions*, if the function $\omega(X_1, \ldots, X_\ell)$ vanishes, if more than k linearly independent vector fields are vertical, and there exist k vertical fields with $\omega(X_1, \ldots, X_\ell) \neq 0$. Of course, we must have $k \leq \min(n - 1, \ell)$.

A tangential vector x at B_0 will be called *spherical*, if it is tangential at the sphere bundle $S(\delta)$ (δ is the distance of a base point from the zero section), i.e. if $dr(x) = 0$.

In T_eB_0 there are the following types of vectors:

(1) one normal vector $A(e)$
(2) $n - 1$ vertical spherical vectors
(3) $d = \dim M$ horizontal vectors (they are not necessarily spherical, which is due to the curvature).

Let $x \in T_eB_0$, $\varepsilon = \|e\|$. By definition of the length of x we have

- $\|x\| = \|K(x)\|$ for x vertical
- $\|x\| = \|\pi_{0*}(x)\|$ for x horizontal

If x is normal, then $\varphi_*(x) = 0$. If x is vertical spherical, then $K\varphi_*(x) = K(\varepsilon_*^{-1}(x)) = \varepsilon^{-1}K(x)$, by sections 3.2.2 and 2.2.10, therefore we get

$$\|\varphi_*(x)\| = \varepsilon^{-1} \tag{20}$$

Lemma 3.2. *If $\omega \in \Omega^{n-1}(S)$ measures k vertical directions, then $\varphi^*(\omega)$ has a pole of order k in the zero section.*

Proof. $\|\varphi^*\omega_e\| = \varepsilon^{-k} \cdot \|\omega_{\varphi(e)}\|$, where $\varepsilon = r(e)$, therefore $\|\varphi^*\omega\| = O(r^{-k})$ because of (20) and (19) in section 3.2.1. □

3.2.5. *Forms with maximal order of poles.*

Example 3.1. A form on S, which measures all vertical directions, is the fiber volume form σ on S. A form Φ on S with the property that $j_m^*\Phi = 0$ measures at most $n-2$ directions; then $\varphi^*\Phi$ on B_0 has a pole of order at most $n-2$ in the zero section.

3.2.6. *Limit integral of forms with not maximal order of poles.*

Lemma 3.3. *Let Φ be a $n-1$–form on B_0 with a pole in the zero section of order at most $n-2$. Then we have*

$$\lim_{\delta\to 0}\int_{S_m(\delta)} \Phi = 0$$

Proof. As $\|\Phi\| = O(r^{-(n-2)})$ and $\mathrm{vol}(S_m(\delta)) = \mathrm{vol}(\mathbb{S}^{n-1}) \cdot \delta^{n-1}$, i.e. $\mathrm{vol}(S_m(\delta)) = O(\delta^{n-1})$, the Lemma follows from the trivial fact, that for a compact Riemannian manifold M of $\dim M = d$ and a d–form ω on M we have

$$|\int_M \omega\,| \le \max\|\omega\| \cdot \mathrm{vol}(M)$$

□

3.3. **Index calculations.**

3.3.1. *The degree of a map.* Let M, N be compact oriented manifolds of same dimension d. Let $f : M \longrightarrow N$ be smooth and $n \in N$ a *regular* value, i.e. $f^{-1}(n)$ contains no critical points, hence $\forall\, m \in f^{-1}(n)\quad T_mf : T_mM \longrightarrow T_nN$ is surjective (therefore bijective) and f is in each $m \in f^{-1}(n)$ étale. In particular $f^{-1}(n)$ may be empty.

The *degree* of f is defined as (see [12])

$$\deg f := \sum_{m\in f^{-1}(n)} \mathrm{sig}\, T_mf$$

where $\mathrm{sig}\, T_mf := \pm 1$, depending on when T_mf respects the orientation or not (Milnor shows that $\deg f$ is independent of $n \in N$).

There is the following equivalent cohomological definition:

Let $\mu \in H^d(M)$ and $\nu \in H^d(N)$ be the fundamental classes, then $f^*(\nu) = \deg f \cdot \mu$.

We prove the equivalence in the following reformulation:

Let v be the volume form of a Riemannian metric on N. By definition of the volume of N the form $\mathrm{vol}(N)^{-1} \cdot v$ lies in the fundamental class of N and we have to show that

$$\deg f = \mathrm{vol}(N)^{-1} \cdot \int_M f^*(v) \tag{$*$}$$

Proof. f is étale in each $m \in f^{-1}(n)$ and the neighbourhood fibers contain the same number of points, say $r = \operatorname{card} f^{-1}(n)$. Therefore f is locally a r–fold covering. Choose $V \subset N$ open, connected, $n \in V$, $U_1, \dots, U_r \subset M$ with $f^{-1}(V) = \bigcup_i U_i$, $f|U_i : U_i \xrightarrow{\sim} V$ diffeomorph. Represent the fundamental class of N by a form ν with support in V. Then we have $\int_M f^*\nu = \sum_i \int_{U_i} (f|U_i)^*\nu = \sum_{m \in f^{-1}(n)} \operatorname{sig} T_m f \cdot \int_V \nu = \deg f$ ☐

3.3.2. *The index of a vector field.* Let $X : M \longrightarrow TM$ be a vector field and let $m \in M$ be an isolated *singularity*: $X_m = 0$.

Let $f : U \to \mathbb{R}^n$ be a local chart, centered at m (i.e. $f(m) = 0$), let $df : TU \to \mathbb{R}^n$ be its derivative and consider the vector field $Y : U' \to \mathbb{R}^n$, $U' := f(U)$, in $\mathbb{R}^n$, $Y := df \circ X \circ f^{-1}$.

Y has an isolated singularity at $0 \in U'$ and on $U' - \{0\}$ we can define $V := Y/\|Y\| : U' - \{0\} \to \mathbb{S}^{n-1}$ (U is assumed to be chosen small enough such that X has no other singularity in U). For a suitable $\delta > 0$ we have $\mathbb{S}^{n-1}(\delta) \subset U' - \{0\}$ and the *index of X at m* is defined by

$$\operatorname{ind}(X, m) := \deg(V|\mathbb{S}^{n-1}(\delta))$$

Milnor shows in [12], that this definition is independent of the chosen chart. By $(*)$ of section 3.3.1 we have

$$(**) \qquad \operatorname{ind}(X, m) = \operatorname{vol}(\mathbb{S}^{n-1})^{-1} \int_{\mathbb{S}^{n-1}(\delta)} V^*(\sigma)$$

where σ is the sphere volume form (see section 1.2.5). Now we have $V = \varphi \circ Y$ and $V^*(\sigma) = Y^*(\sigma/r^n)$ and σ/r^n is closed in $\mathbb{R}^n - \{0\}$. From this results (via Stokes) the independence of δ.

From the invariance of the degree under homotopy results the independence of the index under a homotopy of the vector field, which leaves the isolated singularity fix and does not contribute new singularities in the neighbourhood. In particular, let us note this

Lemma 3.4. *Let $X : U \longrightarrow \mathbb{R}^n$ a vector field and $m \in U$ an isolated singularity. Let $f = id + r^2 \cdot h$ in a neighbourhood of 0 with bounded h. Then*

$$\operatorname{ind}(X, m) = \operatorname{ind}(f \circ X, m)$$

Proof. f is obviously homotopic to the identity. We have to show that f does not vanish outside of 0: $f(x) = x + \|x\|^2 \cdot h(x)$, $\|x\| \le \|f(x)\| + \|x\|^2 \cdot C$, as $\|h\| \le C$ is bounded. Therefore $\|f(x)\| \ge \|x\|(1 - \|x\| \cdot C)$, and for $\|x\|$ sufficiently small we are done. ☐

3.3.3. *Comparison Lemma.*

Lemma 3.5. *Let M be an oriented Riemannian manifold of dimension n, σ the fiber volume form on S, the sphere bundle of tangent unit vectors, and $X : M \to TM$ a vector field on M with an isolated singularity in $m \in M$. Then we have*

$$\operatorname{ind}(X, m) = \operatorname{vol}(\mathbb{S}^{n-1})^{-1} \cdot \lim_{\delta \to 0} \int_{S_m(\delta)} (\varphi \circ X \circ \exp_m)^*(\sigma)$$

Proof. On each fiber S_m we have the canonical volume form σ_m. A simple consideration shows, that the index is exactly calculated, when we replace σ with the form we get from parallel transport of σ_m along the geodesic. The formula follows by taking the $\lim_{\delta \to 0}$.

As chart around m we take the one given by the exponential function (*normal coordinates*). For a suitable $\varepsilon > 0$ we have

$$\exp_m : B^0_m(\varepsilon) \xrightarrow{\sim} U_m(\varepsilon) \subset M$$

a diffeomorphism of the open ε–ball in the fiber $E_m := T_m M$ onto a neighbourhood of m. Choose $f : U_m(\varepsilon) \longrightarrow B^0_m(\varepsilon)$, $f = \exp_m^{-1}$. By 3.3.2, $(**)$ we have with $c_n = \mathrm{vol}(\mathbb{S}^{n-1})$

$$\text{ind}\,(X, m) = c_n^{-1} \cdot \int_{S_m(\delta)} (\varphi_m \circ df \circ X \circ f^{-1})^*(\sigma_m) \tag{21}$$

where we have put $\varphi_m = \varphi | E_m - \{0\}$ (with $0 < \delta < \varepsilon$ suitable).

For $m' \in U_m(\varepsilon)$ let $\tau_{m'} : E_m \xrightarrow{\sim} E_{m'}$ be the isometry given by the parallel transport of vectors along the geodesic from m to m' (see [1, II B.IV.14], or [7, I Prop. 5.2]).

The parallel transport defines a trivialization of E over $U_m(\varepsilon)$:

$$\begin{array}{rccc} \tau : & B^0_m(\varepsilon) \times E_m & \longrightarrow & E|U_m(\varepsilon) \\ & (e, x) & \longmapsto & \tau_{\exp_m e}(x) \end{array}$$

with the inverse being ($y \in E|U_m(\varepsilon)$)

$$\tau^{-1}(y) = (f \circ \pi(y), \tau^{-1}_{\pi(y)}(y))$$

As parallelism conserves the metric the following diagram is commutative:

$$\begin{array}{ccc} \tau : B^0_m(\varepsilon) \times E_m - \{0\} & \longrightarrow & E_0|U_m(\varepsilon) \\ \downarrow{\scriptstyle 1\times\varphi_m} & & \downarrow{\scriptstyle \varphi} \\ \tau : B^0_m(\varepsilon) \times S_m & \longrightarrow & S|U_m(\varepsilon) \end{array}$$

Another trivialization is given by f_*:

$$f_* : E|U_m(\varepsilon) \xrightarrow{\sim} B^0_m(\varepsilon) \times E_m$$
$$f_*(y) = (f \circ \pi(y), T_{\pi(y)} f(y))$$

These two trivializations coincide in the *first infinitesimal neighbourhood* of m, i.e. $\tau \equiv (\exp_m)_* \mod r^2$.

This is clear at the point m, as $T_{0_m} \exp_m : E_m \longrightarrow E_m$ is the identity. For $m' \neq m$ consider in a chart instead of $e = f(m')$, $e \in B^0_m(\varepsilon) - \{0_m\}$, its polar coordinates $t = \|e\| \in \mathbb{R}$, $e_1 = e/t \in S_m$. By [1, II, E.III.6], we have $\forall\, x \in E_m$

$$\tau^{-1}_{\exp_m e}(T_{te_1} \exp_m(x)) = x - \frac{t^2}{6} R(x, e_1)e_1 + O(t^3)$$

We get therefore a comparison of the differential

$$df : E|U_m(\varepsilon) \xrightarrow{f_*} B^0_m(\varepsilon) \times E_m \xrightarrow{pr_2} E_m$$

with

$$h : E|U_m(\varepsilon) \xrightarrow{\tau^{-1}} B^0_m(\varepsilon) \times E_m \xrightarrow{pr_2} E_m$$

and can apply the lemma in 3.3.2

$$\text{ind}\,(df \circ X \circ f^{-1}, 0_m) = \text{ind}\,(h \circ X \circ f^{-1}, 0_m)$$

and (21) becomes

$$\text{ind}\,(X, m) = c_n^{-1} \cdot \int_{S_m(\delta)} (\varphi_m \circ h \circ X \circ f^{-1})^*(\sigma_m) \tag{22}$$

Now by the diagram above we have $\varphi_m \circ h = h \circ \varphi$. We therefore have to compare the forms $h^*(\sigma_m)$ and σ. As $j_m^*(\sigma) = \sigma_m$, we compare $(j_m \circ h)^*(\sigma)$ with σ by looking at $j_m \circ h$ in the trivialization given by τ: we find $\tau^{-1} \circ j_m \circ h \circ \tau : B_m^0(\varepsilon) \times S_m \xrightarrow{pr_2} S_m \hookrightarrow B_m^0(\varepsilon) \times S_m$ is given by $(e, x) \mapsto (0, x)$. Hence $h^*(\sigma_m) \to \sigma$ as $\delta \to 0$, and the lemma follows from (22). □

3.4. **The Chern construction.** In this section we give explicit formulas for the solution of the differential equations in theorem 1.2. This construction goes back to Chern [2].

3.4.1. *Scalar connection and curvature forms.* Let P/M be a principal bundle with structure group G, let ω, Ω be the $\mathfrak{g}$–valued connection resp. curvature forms of a connection on P ($\mathfrak{g}$ =Lie G).

Let $(v_i)_i$ be a basis of the Lie algebra $\mathfrak{g}/\mathbb{R}$.

Let

$$\omega = \sum_i \omega_i \cdot v_i, \quad \Omega = \sum_i \Omega_i \cdot v_i \tag{23}$$

be the components of the forms with respect to this basis. The forms ω_i and Ω_i are ordinary 1– resp. 2–forms on P.

For the form $[\omega, \omega]$ we have (with tangential vectors x, y):

$$[\omega, \omega](x, y) = [\omega(x), \omega(y)] = \sum_{i,j} \omega_i(x) \cdot \omega_j(y) \cdot [v_i, v_j] =$$
$$= \sum_{i<j} \big(\omega_i(x)\omega_j(y) - \omega_j(x)\omega_i(y)\big)[v_i, v_j]$$

As $(\omega_i \wedge \omega_j)(x, y) = \omega_i(x)\omega_j(y) - \omega_j(x)\omega_i(y)$ we get

$$[\omega, \omega](x, y) = \frac{1}{2} \cdot \sum_{i,j} \omega_i \wedge \omega_j \cdot [v_i, v_j] \tag{24}$$

(the sums for $i < j$ and $i > j$ are equal, therefore the factor).

3.4.2. *Structure group $SO(n)$.* Now let the structure group be $G = SO(n)$, its Lie algebra is $\mathfrak{g} = \mathfrak{so}(n) \subset \mathfrak{gl}(n) = \mathrm{End}(\mathbb{R}^n)$, the subspace of skew–symmetric matrices (see [7, chap. X, §2.1]). A basis is $v_{ij} = E_{ij} - E_{ji}$, $1 \le i < j \le n$, where $E_{ij} = (\delta_{ri} \cdot \delta_{sj})_{rs}$ is the matrix with a 1 in the i^{th} row, j^{th} column, and 0 elsewhere.

Then the formulas (23), (24) in section 3.4.1 become with $\omega_{ij} := -\omega_{ji}$ for $i > j$:

$$\omega = \sum_{i<j} \omega_{ij} \cdot v_{ij} = \sum_{i,j} \omega_{ij} \cdot E_{ij}$$

and similar for Ω.

$$[\omega, \omega] = \frac{1}{2} \cdot \sum_{i<j,k<l} \omega_{ij} \wedge \omega_{kl} \cdot [v_{ij}, v_{kl}] = \frac{1}{2} \cdot \sum_{i,j,k,l} \omega_{ij} \wedge \omega_{kl} \cdot [E_{ij}, E_{kl}]$$

The Lie bracket $[E_{ij}, E_{kl}]$ vanishes except for $j = k$ or $i = l$, therefore

$$[\omega, \omega] = \frac{1}{2} \cdot \sum_{i,j,l} \omega_{ij} \wedge \omega_{jl} \cdot E_{il} - \frac{1}{2} \cdot \sum_{i,j,k} \omega_{ij} \wedge \omega_{ki} \cdot E_{kj} = \sum_{i,j,k} \omega_{ij} \wedge \omega_{jk} \cdot E_{ik}$$

From the definition in section 2.2.5 $\Omega = d\omega + [\omega, \omega]$ we obtain for the scalar connection form

$$\Omega_{ik} = d\omega_{ik} + \sum_j \omega_{ij} \wedge \omega_{jk} \tag{25}$$

For its differential one calculates

$$\begin{aligned} d\Omega_{ik} &= \sum_j d\omega_{ij} \wedge \omega_{jk} - \sum_j \omega_{ij} \wedge d\omega_{jk} = \\ &= \sum_j \Omega_{ij} \wedge \omega_{jk} - \sum_j \omega_{ij} \wedge \Omega_{jk} - \\ &\quad - \sum_{j,l} \omega_{il} \wedge \omega_{lj} \wedge \omega_{jk} + \sum_{j,l} \omega_{ij} \wedge \omega_{jl} \wedge \omega_{lk} \end{aligned}$$

and the last two terms cancel – hence

$$d\Omega_{ik} = \sum_j \big(\Omega_{ij} \wedge \omega_{jk} - \Omega_{kj} \wedge \omega_{ji}\big) \tag{26}$$

3.4.3. *Principal bundle over the sphere bundle.* We revisit the situation in 1.2.2 ff., i.e. let E/M be an oriented Riemannian vector bundle, let $\pi_1 : S \longrightarrow M$ be its sphere bundle and $\pi : P \longrightarrow M$ be its principal bundle with structure group $SO(n)$ (see section 3.1.2).

On P let a connection be given by $\omega = (\omega_{ij})$, with curvature form $\Omega = (\Omega_{ij})$.

P is in a natural way also a principal bundle over S !

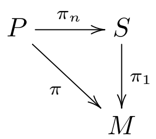
$$\pi_n : P \longrightarrow S$$
$$\pi_n(p) = p(e_n)$$

(where $e_n \in \mathbb{R}^n$ is the n^{th} canonical unit vector).

The structure group of P/S is the subgroup of $SO(n)$ that leaves e_n fix, hence $SO(n-1)$ considered as the subgroup

$$\begin{array}{ccc} SO(n-1) & \subset & SO(n) \\ g & \longmapsto & \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

The following diagram is by definition commutative



Let us determine the vertical vectors for π_n. Since $\pi = \pi_1 \circ \pi_n$ these are vertical for π as well. From the embedding of $SO(n-1) \subset SO(n)$ we can see by the sequence (15) in section 2.2.2 that the vertical space of π_n consists of those vertical vectors for π, for which the last $n-1$ coordinates ω_{in} $\quad (1 \le i \le n-1)$ vanish:

$$\forall\, x \in T_pP \quad \pi_{n*}(x) = 0 \quad \Leftrightarrow \quad \pi_*(x) = 0,\ \omega_{in}(x) = 0 \tag{27}$$

3.4.4. *The forms of* Chern.

Definition 3.5.

$$\Phi_k := \sum_\alpha \varepsilon_\alpha \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge \omega_{\alpha_{2k+1}n} \wedge \cdots \wedge \omega_{\alpha_{n-1}n}$$

$$\Psi_k := 2(k+1) \sum_\alpha \varepsilon_\alpha \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge \Omega_{\alpha_{2k+1}n} \wedge \omega_{\alpha_{2k+2}n} \wedge \cdots \wedge \omega_{\alpha_{n-1}n}$$

where $\alpha \in \mathbf{S}_{n-1}$ runs thru the permutations of $1, 2, \ldots, n-1$, and $\varepsilon_\alpha = \operatorname{sig} \alpha$.

The forms Φ_k are defined for $0 \leq k \leq \left[\frac{n-1}{2}\right]$, the forms Ψ_k for $0 \leq k \leq \left[\frac{n}{2}\right] - 1$, as is obvious from their definition. One sets $\Psi_{-1} = \Psi_{[n/2]} = 0$.

The Φ_k (resp. Ψ_k) are $(n-1)$–forms (resp. n–forms) on P.

By theorem 2.3 for all $g \in SO(n)$ we have $g^*(\Omega) = g^{-1} \cdot \Omega \cdot g$, hence for the scalar curvature forms $(g^*\Omega)_{ij} = \sum_{k,l} g_{ki}\Omega_{kl}g_{lj}$ and similarly (see definition 2.3) $(g^*\omega)_{ij} = \sum_{k,l} g_{ki}\omega_{kl}g_{lj}$.

For $g \in SO(n-1)$ we obtain therefore $(1 \leq i, j \leq n-1)$:

(∗)
$$(g^*\Omega)_{ij} = \sum_{k,l} \Omega_{kl}g_{ki}g_{lj}$$
$$(g^*\omega)_{in} = \sum_{k} \omega_{kn}g_{ki}$$

We will now apply the Lemma 2.2 to the ordinary forms Φ_k, Ψ_k on the principal bundle P/S.

They vanish if applied on vertical vector for π_n: for the factors ω_{in} this is clear by (27), for the Ω_{ij} this is the theorem 2.3. Therefore the first relation (1) in the Lemma 2.2 holds. Moreover by the relations (∗) above and by the definition of the determinant we get for $g \in SO(n-1)$ $\quad g^*(\Phi_k) = \det g \cdot \Phi_k = \Phi_k$, as well as for Ψ_k; the forms are $SO(n-1)$–invariant and (2) in Lemma 2.2 is fulfilled.

There exist therefore uniquely determined forms on S, which pulled back to P are the forms Φ_k, Ψ_k.

In the case of even fiber dimension $n = 2p$ the form Ψ_{p-1} even comes from M, as there are no factors ω_{in}, and we can apply the same reasoning as before, this time to the principal bundle P/M.

3.4.5. *Differential of* Φ_k. We calculate the differential of the forms Φ_k. On this occasion the differential of the Ω_{ij} occurs k–times, it suffices to differentiate one of them due to the permutations; the ω_{in} occur $(n-2k-1)$–times. Hence

$$d\Phi_k = k \cdot \sum_{\alpha} \varepsilon_\alpha \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge d\Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge \omega_{\alpha_{2k+1}n} \wedge \cdots \wedge \omega_{\alpha_{n-1}n} +$$
$$+ (n-2k-1) \cdot \sum_{\alpha} \varepsilon_\alpha \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge d\omega_{\alpha_{2k+1}n} \wedge \cdots \wedge \omega_{\alpha_{n-1}n}$$

Substituting the formulas (26) and (25) here gives

$$d\Phi_k = 2k \cdot \sum_{\alpha,i} \varepsilon_\alpha \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}i} \wedge \omega_{i\alpha_{2k}} \wedge \omega_{\alpha_{2k+1}n} \wedge \cdots \wedge \omega_{\alpha_{n-1}n} -$$
$$- (n-2k-1) \cdot \sum_{\alpha,i} \varepsilon_\alpha \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge \omega_{\alpha_{2k+1}i} \wedge \omega_{in} \wedge \cdots \wedge \omega_{\alpha_{n-1}n} +$$
$$+ (n-2k-1) \cdot \sum_{\alpha} \varepsilon_\alpha \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k+1}n} \wedge \omega_{\alpha_{2k+2}n} \wedge \cdots \wedge \omega_{\alpha_{n-1}n}$$

With Φ_k also $d\Phi_k$ comes from S, therefore vanishes on vertical vectors (with respect to π_n). Thus the sum of terms $i \neq n$ in the first sum cancel versus the second sum. The remaining part sum for $i = n$ is equal to $-\Psi_{k-1}$. The third sum finally is equal to $(n-2k-1)/2(k+1)\Psi_k$, hence

$$d\Phi_k = \frac{n-2k+1}{2(k+1)} \cdot \Psi_k - \Psi_{k-1}$$

This implies that the forms Ψ_k are exact, as

$$\Psi_k = \frac{2(k+1)}{n-2k-1} \cdot (d\Phi_k + \Psi_{k-1})$$

and the assertion follows by recurrence (with $\Psi_{-1} = 0$):

$$\Psi_k = d\theta_k \tag{28}$$

$$\theta_k = \sum_{i=0}^{k} \prod_{l=i}^{k} \frac{2l+2}{n-2l-1} \cdot \Phi_i \tag{29}$$

3.4.6. *The case $n = 2p$.* By definition

$$\Psi_{p-1} = n \sum_{\alpha} \varepsilon_\alpha \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{n-1}n} = \sum_{\lambda} \varepsilon_\lambda \Omega_{\lambda_1\lambda_2} \wedge \cdots \wedge \Omega_{\lambda_{n-1}\lambda_n}$$

where $\lambda \in \mathbf{S}_n$ runs thru the permutations of $1, 2, \ldots, n$. We have already seen in 3.4.4 that this form comes from M. We consider (28) and (29) for $k = p-1$. After division by $2^n p! \pi^p$ (this $\pi \in \mathbb{R}$!) we define

$$\kappa := (2^n p! \pi^p)^{-1} \sum_{\lambda} \varepsilon_\lambda \Omega_{\lambda_1\lambda_2} \wedge \cdots \wedge \Omega_{\lambda_{n-1}\lambda_n}$$

$$\Pi := \pi^{-p} \sum_{i=0}^{p-1} a_i \Phi_i \qquad \text{where } a_i^{-1} := 1 \cdot 3 \cdots (2p-2i-1) 2^{p+i} i!$$

and (28) now reads

$$\kappa = d\Pi \tag{$*$}$$

3.4.7. *The case $n = 2q+1$.* This time we consider (28), (29) for $k = q-1$. As $\Psi_q = 0$ we have $d\Phi_q = -\Psi_{q-1}$, and by (28) $d\theta_{q-1} = \Psi_{q-1}$, hence $\theta_{q-1} + \Phi_q$ is closed, $\theta_{q-1} + \Phi_q = \sum_{i=0}^{q-1} \frac{q \cdots (i+1)}{1 \cdots i} \Phi_i + \Phi_q = \sum_{i=0}^{q} \binom{q}{i} \Phi_i$. Define

$$\kappa := 0$$

$$\Pi := \frac{-1}{2^n q! \pi^q} \sum_{i=0}^{q} \binom{q}{i} \Phi_i$$

and ($*$) is again satisfied.

Note. Chern in [2] also notes the following formula for Π valid in both cases $n = 2p, n = 2q+1$, which is easily verified using the functional equation $\Gamma(x+1) = x\Gamma(x)$ and the special value $\Gamma(\frac{1}{2}) = \pi^{1/2}$ (Γ is the Gamma function):

$$\Pi = \frac{(-1)^n}{2^n \pi^{\frac{n-1}{2}}} \cdot \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{\Gamma(\frac{n-2i+1}{2}) i!} \Phi_i \tag{$**$}$$

The form Π here is Chern's form $(-1)^n \Pi$; the form κ is denoted Ω by Chern.

3.4.8. *The fiber volume form of the sphere bundle.*

Lemma 3.6. *Let σ be the fiber volume form on the sphere bundle S. Then we have*

$$\pi_n^*(\sigma) = (-1)^{n-1} \omega_{1n} \wedge \cdots \wedge \omega_{n-1,n}$$

Proof. We show that both forms coincide at an arbitrary point $p \in P$. We choose the notation of 3.1.5, where the local section $s : U \to P$ now passes thru p: $s(m) = p$, if $p \in P_m$. By definition we have $\pi_n(p) = p(e_n) = s_n(m)$, and from the coordinate representation we get

$$y^i \circ \pi_n(p) = \delta_{in}$$

Furthermore we have

$$\pi_n^*(\kappa^i)_p = y^i \circ K \circ \pi_{n*} : \begin{array}{ccccccc} T_pP & \longrightarrow & T_{\pi_n(p)}S & \xrightarrow{K} & E_m & \xrightarrow{y^i} & \mathbb{R} \\ x & \longmapsto & xe_n & \longmapsto & p\omega_p(x)e_n & \longmapsto & \omega_{in}(x) \end{array}$$

according to the description given to K in 2.2.9 and since

$$p\omega_p(x)e_n = p(\sum_i \omega_{in}(x) \cdot e_i) = \sum_i \omega_{in}(x) \cdot s_i(m)$$

Hence $\pi_n^*(\kappa^i)_p = \omega_{in|p}$. As $\pi_n^*(y^i)_{|p} = \delta_{in}$ from the local expression for σ follows $\pi_n^*(\sigma)_{|p} = (-1)^{n+1}(\omega_{1n} \wedge \cdots \wedge \omega_{n-1,n})_p$, qed.

□

3.4.9. *The solution of the differential equation.* We will now show that the differential forms defined in sections 3.4.6, 3.4.7 are solutions of the differential equations (C) in the theorem 1.2.

Let us first explain how to read the formula ($**$) for Π: the sum consists of a summand Φ (for $i \geq 1$), which contains curvature terms, and of the first summand Φ_0 (up to a factor). Now by definition

$$\Phi_0 = \sum_\alpha \varepsilon_\alpha \omega_{\alpha_1 n} \wedge \cdots \wedge \omega_{\alpha_{n-1} n} = (n-1)!\omega_{1n} \wedge \cdots \wedge \omega_{n-1,n} =$$
$$= (-1)^{n-1}(n-1)!\pi_n^*(\sigma) \qquad \text{by Lemma 3.6}$$

To understand the factor in front of Φ_0, let us remark that $\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2}+1) = \frac{n!}{2^n}\sqrt{\pi}$. Then we obtain

$$\frac{2^n \pi^{(n-1)/2}\Gamma(\frac{n+1}{2})}{(n-1)!} = \frac{n\sqrt{\pi^n}}{\Gamma(\frac{n}{2}+1)} = \frac{2\sqrt{\pi^n}}{\Gamma(\frac{n}{2})} = \mathrm{vol}(\mathbb{S}^{n-1}) =: c_n$$

We now treat the forms Φ and Π as forms on S (see 3.4.4). Then the formula ($**$) in section 3.4.7 becomes

$$\Pi = -c_n^{-1}\sigma + \Phi \tag{30}$$

The first term $c_n^{-1}\sigma$ represents on each fiber the fundamental class. For the second term we have: $j_m^*(\Phi) = 0$. This needs only to be shown for the pulled back form on P and there it results from theorem 2.3 as the curvature terms effectively occur. We treat now κ as form on M and ($*$) in section 3.4.6 becomes

$$\pi_1^*(\kappa) = d\Pi \tag{31}$$

From this it already follows that κ is closed on M. Like before the Lemma 2.2 tells us it suffices to prove this for the pull back on P, but there κ is even closed by ($*$).

We have thus solved the equations (C) in theorem 1.2 by explicitely given differential forms.

3.5. Gauss–Bonnet formula.

Theorem 3.7 (Gauss–Bonnet)**.** *Let M be a compact oriented Riemannian manifold of dimension n. Let κ be the n–form constructed in 3.4 from the* Levi–Civita *connection on M. Then we have*

$$\int_M \kappa = \chi(M)$$

Proof. (Chern)

The Levi–Civita connection $K : TE \to E$, where the vector bundle $E = TM$ is the tangential bundle, belongs canonically to the Riemannian manifold (see [6]). We have seen in 2.2.11 how to associate a connection on the principal bundle belonging to E. The form κ is therefore distinguished in a canonical way by the data given on M (the independence of the cohomology class of κ from the connection can be shown in general – we will not go into this detail).

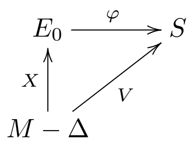
Let $X : M \longrightarrow TM$ be a vector field on M with only isolated singularities (existence will be seen in an instant). In particular the set of singularities

$$\Delta := \{m \in M \mid X_m = 0\}$$

is finite (since M is compact). Around each singularity cut an ε–neighbourhood $U_m(\varepsilon)$ (open ε–ball around m). Let ε be chosen up front sutitably, such that the assumptions made in 3.3.3 apply, and the different $U_m(\varepsilon)$ do not intersect each other. We have an ε–neighbourhood of all singularities

$$\Delta_\varepsilon := \bigcup_{m \in \Delta} U_m(\varepsilon)$$

and $M - \Delta_\varepsilon$ is a compact Riemannian submanifold of same dimension n of M with boundary and canonical orientation. Look at the unit vector field V on $M - \Delta$ (with previous notations, cf. 1.2):



On $U_m(\varepsilon)$ we choose the local chart $f : U_m(\varepsilon) \xrightarrow{\sim} B_m^0(\varepsilon)$, $\quad f = \exp_m^{-1}$. For δ with $0 < \delta < \varepsilon$ we then have

$$\int_{M-\Delta_\delta} \kappa = \int_{M-\Delta_\delta} (\pi_1 \circ V)^*(\kappa) = \int_{M-\Delta_\delta} V^*(\pi_1^*\kappa) = \int_{M-\Delta_\delta} dV^*(\Pi) =$$
$$= - \sum_{m\in\Delta} \int_{\partial U_m(\delta)} V^*\Pi$$

by section 3.4.9 (31) and the formula of Stokes, the minus sign results from the opposite orientation of $\partial U_m(\delta)$ from the orientation of the boundary $\partial(M - \Delta_\delta)$ (an outer normal vector of the former being an inner normal vector of the latter). As $\exp_m$ is orientation faithful, we get further on

$$\int_{\partial U_m(\delta)} V^*\Pi = \int_{S_m(\delta)} (V \circ \exp_m)^*\Pi$$

and with section 3.4.9 (30) finally

$$\int_{M-\Delta_\delta} \kappa = c_n^{-1} \sum_{m\in\Delta} \int_{S_m(\delta)} (\varphi \circ X \circ \exp_m)^*(\sigma) - \sum_{m\in\Delta} \int_{S_m(\delta)} (X \circ \exp_m)^* \varphi^*(\Phi)$$

By 3.2.5 $\varphi^*(\Phi)$ has a pole at most of order $n-2$ in the zero section and $X \circ \exp_m$ does not generate new poles. By 3.2.6 and 3.3.3 we obtain for $\delta \to 0$ the formula

$$\int_M \kappa = \sum_{m\in\Delta} \mathrm{ind}\,(X, m)$$

The sum of the indices is therefore independent of the chosen vector field. It is a theorem of Hopf, that this sum equals the EULER-POINCARÉ characteristic $\chi(M)$. The proof is simple using MORSE theory (cf. [11]).

Let f be a MORSE function on M (which exists by [11, cor. 6.7, p. 36]) and let $X := \mathrm{grad}\, f$ be the gradient field. Locally at a singularity of X (a critical point of f) the function can be written

$$f = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

where λ is the MORSE index of f. Let $g^{ij} := g(dx_i, dx_j)$, where g is the Riemannian metric, then we have $\mathrm{grad}\, f = 2(\sum_{i\le\lambda,j} -x_i g^{ij}\frac{\partial}{\partial x_j} + \sum_{i>\lambda,j} x_i g^{ij}\frac{\partial}{\partial x_j})$ and in the base fields $v^i := \sum_j g^{ij}\frac{\partial}{\partial x_j}$ the field X has coordinates $(-x_1, \dots, -x_\lambda, x_{\lambda+1}, \dots, x_n)$, and therefore $\mathrm{ind}\,(X, m) = (-1)^\lambda$ (λ reflections at hyperplanes, which reverse λ–times the orientation). When c_λ is the number of $m \in \Delta$ with MORSE index λ, we have

$$\sum_{m\in\Delta} \mathrm{ind}\,(X, m) = \sum_\lambda (-1)^\lambda c_\lambda = \chi(M)$$

by the MORSE inequality [11, theorem 5.2].

This completes the proof for GAUSS–BONNET. □

3.6. **Intersection numbers.** We define the intersection number $A \circ B \in \mathbb{Z}$ of two submanifolds A, B with transversal intersection in M (precise assumptions follow). This intersection number can be calculated by integration over differential forms associated to the submanifolds by the mechanism in chapter 1. To this end we use a lemma by John MILNOR [13].

3.6.1. *Local and global intersection numbers in the transversal case.* Let M be a compact oriented Riemannian manifold of dimension n; $A, B \subset M$ are closed submanifolds with $r = \dim A$, $s = \dim B$, $r + s = n$, we assume them to be oriented.

In section 1.2.6 we have associated differential forms $\tau_A \in \Omega^s(M), \tau_B \in \Omega^r(M)$ to them. We start by the assumption that the intersection $A \cap B$ is *transversal*. The normal bundles $NA, NB \subset TM$ are oriented vector bundles by $TA \oplus NA = TM|A$ (as oriented sum in this order), analogously for B. For $m \in A \cap B$ the composition $T_mB \longrightarrow T_mM \longrightarrow N_mA$ is an isomorphism because of the transversality of the intersection.

Definition 3.6. [13, §6.1] The *local intersection number* is defined as

$$(A \circ B)_m := \pm 1$$

according as to $T_mB \xrightarrow{\sim} N_mA$ respects $(+1)$ or not (-1) the orientation.

The (global) intersection number is defined by

$$A \circ B = \sum_{m\in A\cap B} (A \circ B)_m$$

MILNOR in [13] proves by a MAYER–VIETORIS argument, which we take for granted here, the following

Lemma 3.8 (MILNOR). *Let $i : B \hookrightarrow M$ be the inclusion. Then we have*

$$\begin{array}{rccc} i^* : & H^s(M) & \longrightarrow & H^s(B) \\ & i^*(\gamma_A) & = & A \circ B.\mu_B \end{array}$$

where μ_B denotes the fundamental class of B.

Note. Actually MILNOR proves the lemma in its homological variant.

Let $\psi : H_0(A) \longrightarrow H_s(M, M - A)$ be defined by THOM isomorphism and excision. Let $[B]$ be the homological fundamental class. MILNOR's lemma then says: If $g : H_s(B) \to H_s(M) \to H_s(M, M - A)$ is the homological sequence for the inclusions $B \to M \to (M, M - A)$, then $g([B]) = (A \circ B).\psi(1)$.

The cohomological variant is obtained from there like this: the THOM isomorphism ([8]) is $H^0(A) \xrightarrow{\sim} H^s(NA, NA_0)$. As in 1.2.6 we obtain by excision and $\exp : B(NA) \longrightarrow M$ a map $\psi : H^0(A) \longrightarrow H^s(M)$ and the class γ_A is by definition the image of 1: $\psi(1) = \gamma_A$. The corresponding map to g is given by i^*, and by the lemma we have $\mu_B = (A \circ B) i^* \gamma_A$, qed.

Corollary 3.9. *Under the previous assumptions on A, B, M we have*

$$A \circ B = \int_B \tau_A = \int_M \tau_A \wedge \tau_B$$

Proof. This is trivial now, the first equality results from MILNOR's lemma by integration over B with the theorem 1.2.6, the second equality is the POINCARE duality (10) in section 1.1.3 (cup–cap relation). □

3.6.2. *Non-transversal intersection. Self intersection.*

Definition 3.7. For A, B as before, but with not necessarily transversal intersection we define

$$A \circ B = \int_M \tau_A \wedge \tau_B$$

Concluding, we consider the *self intersection* of M. It is defined by $\Delta \circ \Delta$, where $\Delta \subset M \times M$ is the diagonal.

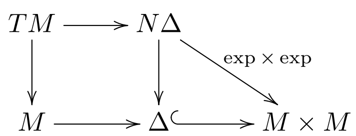
Theorem 3.10.

$$\Delta \circ \Delta = \chi(M)$$

Proof. As we will see, this is only a variant of the GAUSS–BONNET formula.

By definition we have $\Delta \circ \Delta = \int_{M \times M} \tau_\Delta \wedge \tau_\Delta = \int_\Delta \tau_\Delta$.

τ_Δ lies in the THOM class of the normal bundle of Δ in $M \times M$, which is identical with the tangential bundle up to the sign $(-1)^n$ by the choice of the orientation. Furthermore $\Delta \simeq M$ and we have to understand that τ_Δ corresponds to κ. But this follows from the diagram



By section 3.5 we have $\Delta \circ \Delta = \int_\Delta \tau_\Delta = (-1)^n \int_M \kappa = (-1)^n \chi(M)$. For uneven n we have $\chi(M) = 0$ (see 3.4.7), qed. □

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