ON THETA FUNCTIONS

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ABSTRACT. Definition and properties of JACOBI's ϑ–function.

PREFACE

In this paper I present the foundation of JACOBI's ϑ–functions, based on his *Notices sur les fonctions elliptiques* [3, vol. I, 7.] and his lecture *Theorie der elliptischen Functionen* [3, vol. I, 19.]. I derive all his ϑ–relations, in particular his *merkwürdige Relation* of *theta–constants*

$$\vartheta_{00}^4(0,\tau) = \vartheta_{01}^4(0,\tau) + \vartheta_{10}^4(0,\tau)$$

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1. JACOBI'S THETA SERIES

The source of JACOBI's ϑ–functions is in his *Notices sur les fonctions elliptiques* [3, vol. I, 7.] (CRELLE, 1828), as well as his *Fundamenta nova theoriae functionum ellipticarum* ([3, vol. I, 4.], 1829).

Later JACOBI reversed the development and started with the theta series to derive the theory of *elliptic* functions in his lecture *Theorie der elliptischen Functionen, aus den Eigenschaften der Thetareihen abgeleitet* prepared by BORCHARDT in 1838 on behalf of JACOBI [3, vol. I, 19.]. We will take a rapid walk through the first part of JACOBI's lecture. For his notations and comparison with later authors see the section 5.

Let $\mathbb{D} = \{x \in \mathbb{C} \mid |x| < 1\}$ be the open unit disk in $\mathbb{C}$. The complex line $\mathbb{C}$ is the universal covering of $\mathbb{C}^\times = \mathbb{C} - \{0\}$ via the exponential map

$$0 \to \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\mathbf{e}} \mathbb{C}^\times \to 1$$

where $t = \mathbf{e}(x) = \exp(2\pi i x) \in \mathbb{C}^\times$ for $x \in \mathbb{C}$. In the following diagram

$$\begin{array}{ccc} \mathbb{H} & \hookrightarrow & \mathbb{C} \\ \downarrow{\scriptstyle \mathbf{e}} & & \downarrow{\scriptstyle \mathbf{e}} \\ \mathbb{D}^\times & \hookrightarrow & \mathbb{C}^\times \end{array}$$

$\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\} = \mathbf{e}^{-1}(\mathbb{D}^\times)$, the upper half plane, is the universal covering of the puntured disk $\mathbb{D}^\times = \mathbb{D} - \{0\}$. Variables are denoted $x \in \mathbb{C}$ and $t = \mathbf{e}(x) \in \mathbb{C}^\times$, resp. $\tau \in \mathbb{H}$ and (*sic!*) $q = \mathbf{e}(\tau/2) \in \mathbb{D}^\times$.

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2. Definitions and properties

Throughout this note I will use the following definition for ϑ

Definition 2.1. Let $\vartheta : \mathbb{C} \times \mathbb{H} \longrightarrow \mathbb{C}$ be given by:

$$\vartheta(x, \tau) = \sum_{n\in\mathbb{Z}} \mathbf{e}(n^2\tau/2 + nx) = \sum_{n\in\mathbb{Z}} q^{n^2} \cdot t^n$$

Remark. In Jacobi's notation $\vartheta_3(\pi x, q) = \vartheta(x, \tau)$, $q = \mathbf{e}(\tau/2)$, see section 5.

Proposition 2.1. *The series for ϑ converges absolutely and uniformly on compact subsets, defining an analytic function*

$$\begin{aligned} \vartheta : \mathbb{C} \times \mathbb{H} &\longrightarrow \mathbb{C} \\ (x, \tau) &\longmapsto \vartheta(x, \tau) \end{aligned}$$

Proof. Let $K \subset \mathbb{C} \times \mathbb{H}$ be compact, then there is $m, M \in \mathbb{R}, m, M > 0$ such that for all $(x, \tau) \in K$, $\operatorname{Im} \tau \geq m$ and $-M \leq \operatorname{Im} x \leq M$. Now let $q_0 = e^{-\pi m}$ and $t_0 = e^{2\pi M}$, then $|q| = |\mathbf{e}(\tau/2)| = \exp(-\pi \operatorname{Im} \tau) \leq q_0$ and $t_0^{-1} \leq |t| \leq t_0$. We then have

$$|\mathbf{e}(\frac{\tau}{2}n^2 + nx)| = |q|^{n^2} \cdot |t|^n \leq \begin{cases} q_0^{n^2} \cdot t_0^n & n \geq 0 \\ q_0^{n^2} \cdot t_0^{-n} & n < 0 \end{cases}$$

Take an integer $n_0 > \frac{2M}{m}$, then $q_1 = q_0^{n_0} t_0 < 1$ and for $n \geq n_0$ we have $q_0^n t_0 \leq q_1$, hence $q_0^{n^2} \cdot t_0^n = (q_0^n t_0)^n \leq q_1^n$. Similarly for $n \leq -n_0$, since $-n \geq n_0$, $q_0^{n^2} \cdot t_0^{-n} \leq q_1^{-n}$. Hence, the series is majorized by a geometric series on the compact set K. □

This proof of convergence can easily be adopted to the situation where we sum over a shifted set $a + \mathbb{Z}$, $a \in \mathbb{C}$, instead of $\mathbb{Z}$ in the sum defining ϑ, leading to the

Definition 2.2. For $a, b \in \mathbb{C}$ the *shifted* ϑ is defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (x, \tau) = \sum_{n\in\mathbb{Z}} \mathbf{e}\big(\frac{\tau}{2}(n + a)^2 + (n + a)(x + b)\big) = \sum_{n\in a+\mathbb{Z}} \mathbf{e}\big(\frac{\tau}{2}n^2 + n(x + b)\big)$$

So, in particular, $\vartheta(x, \tau) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (x, \tau)$ and $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (x, \tau) = \vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (x + b, \tau)$.

Direct calculation yields the equation

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (x, \tau) = \mathbf{e}\big(\frac{\tau}{2}a^2 + a(x + b)\big) \cdot \vartheta(x + a\tau + b, \tau), \tag{1}$$

which implies

$$\vartheta \begin{bmatrix} a + m \\ b + n \end{bmatrix} (x, \tau) = \mathbf{e}(an) \cdot \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (x, \tau) \qquad \text{for } m, n \in \mathbb{Z}, \tag{2}$$

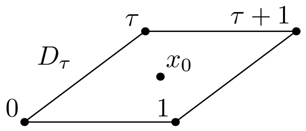
and in particular

$$\vartheta(x + 1, \tau) = \vartheta(x, \tau), \tag{3}$$

$$\vartheta(x + \tau, \tau) = \mathbf{e}(-\frac{\tau}{2} - x) \cdot \vartheta(x, \tau) = q^{-1}t^{-1}\vartheta(x, \tau). \tag{4}$$

We note that ϑ has a zero at $x_0 = (1 + \tau)/2$, as

$$\vartheta(x_0, \tau) = \sum_{n\in\mathbb{Z}} \mathbf{e}(\frac{\tau}{2}n^2 + n\frac{1 + \tau}{2}) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n+1)} = 0.$$



It is in fact the only zero in a fundamental domain D_τ of the lattice $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z} \subset \mathbb{C}$. This follows from the residue theorem: the number of zeros with multiplicity is

$$\sum_{x \in D_\tau} \operatorname{ord}_x \vartheta(x,\tau) = \sum_{x \in D_\tau} Res(\frac{\vartheta'}{\vartheta}, x) = \frac{1}{2\pi i} \int_\gamma \frac{\vartheta'(x,\tau)}{\vartheta(x,\tau)} dx$$

where $\gamma = \partial D_\tau$ is the boundary of D_τ. We evaluate the integral over the path γ:

$$\int_\gamma \frac{\vartheta'(x,\tau)}{\vartheta(x,\tau)} dx = \int_0^1 \Big(\frac{\vartheta'(x,\tau)}{\vartheta(x,\tau)} - \frac{\vartheta'(x+\tau,\tau)}{\vartheta(x+\tau,\tau)} \Big) dx + \int_0^\tau \Big(\frac{\vartheta'(x+1,\tau)}{\vartheta(x+1,\tau)} - \frac{\vartheta'(x,\tau)}{\vartheta(x,\tau)} \Big) dx$$

and the last integral is $= 0$ because of the periodicity (3). The logarithmic derivative of (4) gives the relation

$$\frac{\vartheta'(x+\tau,\tau)}{\vartheta(x+\tau,\tau)} = -2\pi i + \frac{\vartheta'(x,\tau)}{\vartheta(x,\tau)}$$

and hence $\sum \operatorname{ord}_x \vartheta(x,\tau) = 1$, so there is exactly one *simple* zero of $\vartheta(x,\tau)$ at $x_0 \in D_\tau$. □

For $a, b \in \frac{1}{2}\mathbb{Z}$ the following special notation is used.

Definition 2.3. For $a, b \in \{0, 1\}$ define $\vartheta_{ab} = \vartheta \begin{bmatrix} a/2 \\ b/2 \end{bmatrix}$.

By (1) this amounts to

$$\vartheta_{00}(x,\tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}\big(\frac{\tau}{2} n^2 + nx\big) = \vartheta(x,\tau)$$

$$\vartheta_{01}(x,\tau) = \sum_{n \in \mathbb{Z}} (-1)^n \mathbf{e}\big(\frac{\tau}{2} n^2 + nx\big) = \vartheta(x + \frac{1}{2}, \tau)$$

$$\vartheta_{10}(x,\tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}\big(\frac{\tau}{2}(n+\frac{1}{2})^2 + (n+\frac{1}{2})x\big) = \mathbf{e}(\frac{\tau}{8})\mathbf{e}(\frac{x}{2})\vartheta(x + \frac{\tau}{2}, \tau)$$

$$\vartheta_{11}(x,\tau) = \sum_{n \in \mathbb{Z}} i(-1)^n \mathbf{e}\big(\frac{\tau}{2}(n+\frac{1}{2})^2 + (n+\frac{1}{2})x\big) = i\mathbf{e}(\frac{\tau}{8})\mathbf{e}(\frac{x}{2})\vartheta(x + \frac{1+\tau}{2}, \tau)$$

We list a table of half period values[1] $\vartheta_{ab}(x+\lambda)$ for a $\lambda \in \frac{1}{2}\Lambda_\tau$ and where $\varepsilon = \varepsilon(x,\tau) = \mathbf{e}(-\frac{x}{2} - \frac{\tau}{8})$ is an exponential factor:

$$\begin{array}{lll}
\vartheta_{00}(x+\frac{1}{2}) = \vartheta_{01}(x) & \vartheta_{00}(x+\frac{\tau}{2}) = \varepsilon\vartheta_{10}(x) & \vartheta_{00}(x+\frac{1+\tau}{2}) = -i\varepsilon\vartheta_{11}(x) \\
\vartheta_{01}(x+\frac{1}{2}) = \vartheta_{00}(x) & \vartheta_{01}(x+\frac{\tau}{2}) = -i\varepsilon\vartheta_{11}(x) & \vartheta_{01}(x+\frac{1+\tau}{2}) = \varepsilon\vartheta_{10}(x) \\
\vartheta_{10}(x+\frac{1}{2}) = \vartheta_{11}(x) & \vartheta_{10}(x+\frac{\tau}{2}) = \varepsilon\vartheta_{00}(x) & \vartheta_{10}(x+\frac{1+\tau}{2}) = -i\varepsilon\vartheta_{01}(x) \\
\vartheta_{11}(x+\frac{1}{2}) = -\vartheta_{10}(x) & \vartheta_{11}(x+\frac{\tau}{2}) = -i\varepsilon\vartheta_{01}(x) & \vartheta_{11}(x+\frac{1+\tau}{2}) = -\varepsilon\vartheta_{00}(x)
\end{array}$$

these follow from the definitions and (3) and (4) (cf. [3, vol. I, 19., (2.), p. 502]). For completeness we also list the equations corresponding to (3) and (4):

$$\begin{array}{ll}
\vartheta_{01}(x+1) = \vartheta_{01}(x) & \vartheta_{01}(x+\tau) = -q^{-1}t^{-1}\vartheta_{01}(x) \\
\vartheta_{10}(x+1) = -\vartheta_{10}(x) & \vartheta_{10}(x+\tau) = q^{-1}t^{-1}\vartheta_{10}(x) \\
\vartheta_{11}(x+1) = -\vartheta_{11}(x) & \vartheta_{11}(x+\tau) = -q^{-1}t^{-1}\vartheta_{11}(x)
\end{array}$$

[1]dropping τ from the notation

3. Deriving a remarkable relation

In his 1838 lecture Jacobi proceeds to derive several formulas between sums of products of four ϑ–series. To lighten the notation, let us agree that $\vartheta(x) = \vartheta(x_1)\vartheta(x_2)\vartheta(x_3)\vartheta(x_4)$ for vectors $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ and similarly for the other thetas.

Jacobi considers the linear reflection at the hyperplane $x_1 - x_2 - x_3 - x_4 = 0$

$$\mathbb{C}^4 \longrightarrow \mathbb{C}^4$$
$$x \longmapsto x' = x \cdot A$$

given by the matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

which satisfies $A = {}^tA$ and $A^2 = 1$. His *Fundamentalsatz* [3, vol. I, 19.2.,(11.)] is the relation (5) in the next theorem:

Theorem 3.1.

$$\vartheta_{00}(x) + \vartheta_{10}(x) = \vartheta_{00}(x') + \vartheta_{10}(x') \tag{5}$$
$$\vartheta_{00}(x) - \vartheta_{10}(x) = \vartheta_{01}(x') + \vartheta_{11}(x') \tag{6}$$
$$\vartheta_{01}(x) + \vartheta_{11}(x) = \vartheta_{00}(x') - \vartheta_{10}(x') \tag{7}$$
$$\vartheta_{01}(x) - \vartheta_{11}(x) = \vartheta_{01}(x') - \vartheta_{11}(x') \tag{8}$$

Proof. Jacobi's reasoning rests on the observation that for $x, y \in \mathbb{C}^4$ the bilinear form $x \cdot {}^ty = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ is invariant under the involution $x \mapsto x'$:

$$x' \cdot {}^ty' = x \cdot A \cdot {}^t(y \cdot A) = x \cdot A \cdot {}^tA \cdot {}^ty = x \cdot {}^ty$$

With our convention for $x \in \mathbb{C}^4$ we note that by definition

$$\vartheta_{00}(x) = \sum_{n\in\mathbb{Z}^4} \mathbf{e}(\frac{\tau}{2} n \cdot {}^tn + n \cdot {}^tx) \qquad \vartheta_{10}(x) = \sum_{n\in(\frac{1}{2}+\mathbb{Z})^4} \mathbf{e}(\frac{\tau}{2} n \cdot {}^tn + n \cdot {}^tx)$$

Jacobi writes down the formulas relating $n = (a, b, c, d)$ and $n' = (a', b', c', d')$

$$a' = \frac{1}{2}(a + b + c + d) \qquad b' = \frac{1}{2}(a + b - c - d)$$
$$c' = \frac{1}{2}(a - b + c - d) \qquad d' = \frac{1}{2}(a - b - c + d)$$

and for $n \in \mathbb{Z}^4 \cup (\frac{1}{2} + \mathbb{Z})^4$ he emphasizes that the numbers

$$a' + b' = a + b \qquad a' + c' = a + c \qquad a' + d' = a + d$$

are integers, which implies $n' \in \mathbb{Z}^4 \cup (\frac{1}{2} + \mathbb{Z})^4$, hence the involution induces a bijection $\mathbb{Z}^4 \cup (\frac{1}{2} + \mathbb{Z})^4 \xrightarrow{\sim} \mathbb{Z}^4 \cup (\frac{1}{2} + \mathbb{Z})^4$ on the index set. Together with the invariances $n \cdot {}^tn = n' \cdot {}^tn'$, $n \cdot {}^tx = n' \cdot {}^tx'$, the relation (5) becomes obvious.

Applying (5) to $(x_1 + 1, x_2, x_3, x_4)' = (x_1' + \frac{1}{2}, x_2' + \frac{1}{2}, x_3' + \frac{1}{2}, x_4' + \frac{1}{2})$ and making use of the table of half period values we get (6). (7) is identical with (6), by interchanging x with x', as $x'' = x$. (8) is obtained from (6) by applying the reverse operation $(x_1 + \frac{1}{2}, x_2 + \frac{1}{2}, x_3 + \frac{1}{2}, x_4 + \frac{1}{2})' = (x_1' + 1, x_2', x_3', x_4')$. □

The system of equations (5)–(8) can be combined into one vector equation

$$\big(\vartheta_{00}(x), \vartheta_{01}(x), \vartheta_{10}(x), \vartheta_{11}(x)\big)' = \big(\vartheta_{00}(x'), \vartheta_{01}(x'), \vartheta_{10}(x'), \vartheta_{11}(x')\big) \tag{9}$$

Remark. Of course, (9) and the system of equations (5)–(8) are equivalent. The vector components of (9) correspond to the equations (R2)–(R5) of MUMFORD in [5, I, §5], whereas (5)–(8) are identical to JACOBI's table (A.): (1.)–(4.) in [3, vol. I, 19., p. 507].

In the sequel JACOBI substitutes various different vectors into (9). I skip some of them and only list the outcome for the vector $(x, x, y, y)' = (x + y, x - y, 0, 0)$

$$\begin{aligned}\vartheta_{00}(x+y)\vartheta_{00}(x-y)\vartheta_{00}^2(0) &= \vartheta_{00}^2(x)\vartheta_{00}^2(y) + \vartheta_{11}^2(x)\vartheta_{11}^2(y) = \\ &= \vartheta_{01}^2(x)\vartheta_{01}^2(y) + \vartheta_{10}^2(x)\vartheta_{10}^2(y) \\ \vartheta_{01}(x+y)\vartheta_{01}(x-y)\vartheta_{01}^2(0) &= \vartheta_{00}^2(x)\vartheta_{00}^2(y) - \vartheta_{10}^2(x)\vartheta_{10}^2(y) = \\ &= \vartheta_{01}^2(x)\vartheta_{01}^2(y) - \vartheta_{11}^2(x)\vartheta_{11}^2(y) \\ \vartheta_{10}(x+y)\vartheta_{10}(x-y)\vartheta_{10}^2(0) &= \vartheta_{00}^2(x)\vartheta_{00}^2(y) - \vartheta_{01}^2(x)\vartheta_{01}^2(y) = \\ &= \vartheta_{10}^2(x)\vartheta_{10}^2(y) - \vartheta_{11}^2(x)\vartheta_{11}^2(y)\end{aligned}$$

For $y = x$ in particular

$$\vartheta_{00}(2x)\vartheta_{00}^3(0) = \vartheta_{00}^4(x) + \vartheta_{11}^4(x) = \vartheta_{01}^4(x) + \vartheta_{10}^4(x)$$

whereas for $y = 0$ it gives $\vartheta_{00}^2(x)\vartheta_{00}^2(0) = \vartheta_{01}^2(x)\vartheta_{01}^2(0)+\vartheta_{10}^2(x)\vartheta_{10}^2(0)$. Substituting $x \mapsto x + \frac{1}{2} + \frac{\tau}{2}$ yields $-\varepsilon^2\vartheta_{11}^2(x)\vartheta_{00}^2(0) = \varepsilon^2\vartheta_{10}^2(x)\vartheta_{01}^2(0) - \varepsilon^2\vartheta_{01}^2(x)\vartheta_{10}^2(0)$, hence

$$\begin{aligned}\vartheta_{00}^2(x)\vartheta_{00}^2(0) &= \vartheta_{01}^2(x)\vartheta_{01}^2(0) + \vartheta_{10}^2(x)\vartheta_{10}^2(0) \\ \vartheta_{11}^2(x)\vartheta_{00}^2(0) &= \vartheta_{01}^2(x)\vartheta_{10}^2(0) - \vartheta_{10}^2(x)\vartheta_{01}^2(0)\end{aligned}$$

Finally for $x = 0$ we obtain the *remarkable relation* (in JACOBI [3, vol. I, 19., (E.) p. 511] *die merkwürdige Relation*) for the *Theta–Nullwerte*

$$\vartheta_{00}^4(0,\tau) = \vartheta_{01}^4(0,\tau) + \vartheta_{10}^4(0,\tau) \tag{10}$$

i.e.

$$(1+2q+2q^4+2q^9+\dots)^4 = (1-2q+2q^4-2q^9+\dots)^4+16q(1+q^{1\cdot 2}+q^{2\cdot 3}+q^{3\cdot 4}+\dots)^4$$

4. VARIATION WITH THE *module* τ

In the previous section we have kept the *module* τ fixed (and sometimes dropped it from the notation). We are now proving the behaviour of ϑ with respect to variation of τ.

We start by looking at $\vartheta(x, \tau + 1)$.

$$\begin{aligned}\vartheta(x,\tau+1) &= \sum \mathbf{e}(\frac{\tau+1}{2}n^2 + nx) = \sum \mathbf{e}(\frac{\tau}{2}n^2 + \frac{1}{2}n^2 + nx) = \\ &= \sum (-1)^n \mathbf{e}(\frac{\tau}{2}n^2 + nx) = \vartheta_{01}(x,\tau) = \vartheta(x+1/2,\tau)\end{aligned}$$

where we used $(-1)^{n^2} = (-1)^n$. In particular, $\vartheta(x, \tau + 2) = \vartheta(x, \tau)$.

Next we are going to look at $\vartheta(x, -1/\tau)$. JACOBI states in [3, vol. I, 7., p. 264] the formula

$$\mathrm{H}(ix,q) = i\sqrt{\frac{K}{K'}}\exp\left(\frac{Kxx}{\pi K'}\right)\mathrm{H}\left(\frac{Kx}{K'},q'\right)$$

where $q = \exp(\frac{-\pi K'}{K})$ and $q' = \exp(\frac{-\pi K}{K'})$. With $\tau = iK'/K$ (such that $q = \mathbf{e}(\tau/2)$) this can be rewritten (see section 5) in our notation as

$$\vartheta_{11}(x,\tau) = i\sqrt{i/\tau}\exp(-\pi i x^2/\tau)\vartheta_{11}(x/\tau,-1/\tau)$$

This can be transformed into the following set of equivalent equations:

Theorem 4.1.

$$\begin{aligned}\vartheta(x,-1/\tau) &= \sqrt{\tau/i}\,\mathbf{e}(x^2\tau/2)\vartheta(x\tau,\tau)\\ \vartheta_{01}(x,-1/\tau) &= \sqrt{\tau/i}\,\mathbf{e}(x^2\tau/2)\vartheta_{10}(x\tau,\tau)\\ \vartheta_{10}(x,-1/\tau) &= \sqrt{\tau/i}\,\mathbf{e}(x^2\tau/2)\vartheta_{01}(x\tau,\tau)\\ \vartheta_{11}(x,-1/\tau) &= -i\sqrt{\tau/i}\,\mathbf{e}(x^2\tau/2)\vartheta_{11}(x\tau,\tau)\end{aligned}$$

Proof. The equivalence follows from the table of half period values. We are going to prove the first one.

Recall some FOURIER transforms: $\exp(-\pi x^2)$ is its own transform, hence, for $t>0$, $g(x)=\exp(-\pi t x^2)$ has the transform $\widehat{g}(x)=\frac{1}{\sqrt{t}}\exp(-\frac{\pi}{t}x^2)$ and for $h(x)=g(x+a)$ we get $\widehat{h}(x)=\int g(t+a)\mathbf{e}(tx)dt=\int g(t)\mathbf{e}((t-a)x)dt=\widehat{g}(x)\mathbf{e}(-ax)$.

The POISSON formula $\sum h(n)=\sum \widehat{h}(n)$ now yields the equation

$$\sum_n \exp\big(-\pi t(n+a)^2\big)=\frac{1}{\sqrt{t}}\sum_n \exp\big(-\frac{\pi}{t}n^2\big)\mathbf{e}(-an) \tag{11}$$

Now by (1) $\vartheta\begin{bmatrix}a\\0\end{bmatrix}(0,\tau)=\mathbf{e}(\frac{\tau}{2}a^2)\vartheta(a\tau,\tau)=\sum\mathbf{e}\big(\frac{\tau}{2}(n+a)^2\big)$, which is the left hand side of our POISSON equation (11) for $\tau=ti$, whereas the sum on the right hand side is $\sum\mathbf{e}\big(-\frac{n^2}{2\tau}-an\big)=\vartheta(-a,-1/\tau)=\vartheta(a,-1/\tau)$ and equation (11) reads

$$\mathbf{e}(\frac{\tau}{2}a^2)\vartheta(a\tau,\tau)=\sqrt{i/\tau}\,\vartheta(a,-1/\tau)$$

which, by analytic continuation, holds for all $\tau\in\mathbb{H}$. □

5. Notation for theta functions by different authors

The notation in the literature varies. JACOBI himself used different notations for his ϑ–functions at various times. In his *Notices sur les fonctions elliptiques* in 1828 in Crelle's Journal [3, vol. I, 7.], JACOBI introduced the notation Η (Eta) and Θ (Theta) (loc.cit. p. 256) for the numerator resp. denominator of his *sinus amplitudinis*

$$\begin{aligned}\sin\mathrm{am}\frac{2Kx}{\pi}&=\frac{1}{\sqrt{k}}\frac{\mathrm{H}(x)}{\Theta(x)}\\ \mathrm{H}(x)&=2\sqrt[4]{q}\sin x-2\sqrt[4]{q^9}\sin 3x+2\sqrt[4]{q^{25}}\sin 5x-2\sqrt[4]{q^{49}}\sin 7x+\dots\\ \Theta(x)&=1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\dots\end{aligned}$$

A year later in his book *Fundamenta Nova Theoriae Functionum Ellipticarum* [3, vol. I, 4., p. 198, p. 224] he used $H(\frac{2Kx}{\pi})=\mathrm{H}(x)$ and $\Theta(\frac{2Kx}{\pi})=\Theta(x)$ instead (see the remarks of WEIERSTRASS on p. 542). In the 1838 lecture they are called $\vartheta_1(x)=\mathrm{H}(x)$ resp. $\vartheta(x)=\Theta(x)$, where his definitions are as follows:

For $q\in\mathbb{C}^\times$ such that $|q|<1$ JACOBI defines (loc.cit. p. 501):

$$\vartheta(x,q)=\sum(-1)^\nu q^{\nu^2}e^{2\nu xi}\qquad \vartheta_1(x,q)=-\sum i^{2\nu+1}q^{\frac14(2\nu+1)^2}e^{(2\nu+1)xi}$$
$$\vartheta_2(x,q)=\sum q^{\frac14(2\nu+1)^2}e^{(2\nu+1)xi}\qquad \vartheta_3(x,q)=\sum q^{\nu^2}e^{2\nu xi}$$

Later authors introduced other fashions, like θ versus ϑ, and ever changing subscripts. Here is a translation table of notations by selected mathematicians.

JACOBI Notices, 1828	$\Theta(\pi x)$	$\mathrm{H}(\pi x)$		
JACOBI Fundamenta nova, 1829	$\mathit{\Theta}(2Kx)$	$H(2Kx)$		
JACOBI Lecture notes, 1838	$\vartheta(\pi x, q)$	$\vartheta_1(\pi x, q)$	$\vartheta_2(\pi x, q)$	$\vartheta_3(\pi x, q)$
WEIERSTRASS	$\vartheta_0(x\|\tau)$	$\vartheta_1(x\|\tau)$	$\vartheta_2(x\|\tau)$	$\vartheta_3(x\|\tau)$
HERMITE	$\theta_{0,1}(x, \tau)$	$-i\theta_{1,1}(x, \tau)$	$\theta_{1,0}(x, \tau)$	$\theta_{0,0}(x, \tau)$
C. JORDAN	$\theta_2(x, \tau)$	$\theta(x, \tau)$	$\theta_1(x, \tau)$	$\theta_3(x, \tau)$
H. CARTAN	$\vartheta_0(x, \tau)$	$\vartheta_1(x, \tau)$		
MUMFORD	$\vartheta_{01}(x, \tau)$	$-\vartheta_{11}(x, \tau)$	$\vartheta_{10}(x, \tau)$	$\vartheta_{00}(x, \tau)$

WEIERSTRASS in *Einführung der Thetafunctionen* [6, §34.] describes the relation to JACOBI and HERMITE, who defined $\theta_{\mu,\nu}(x) = \sum_m (-1)^{m\nu} \mathbf{e}(\frac{\tau}{2}(m+\frac{\mu}{2})^2 + (m+\frac{\mu}{2})x)$. CARTAN follows WEIERSTRASS in [1, chap. V, ex. 3]. JORDAN introduced *Les fonctions* $\theta(x,\tau)$, $\theta_1(x,\tau)$, $\theta_2(x,\tau)$, $\theta_3(x,\tau)$ (V, n^o 426) in *Fonctions elliptiques* [4, chap. VII] and relates their difference to WEIERSTRASS' notation in n^o 427. CHANDRASEKHARAN uses this notation in [2, V, §8] of JORDAN's, as does WEIL in [7, chap. IV, §8] for $\theta(\zeta, \tau)$.

I have chosen MUMFORD's notation in [5].

References

[1] Henri Cartan, *Elementare Theorie der Analytische Funktionen einer oder mehrerer komplexen Veränderlichen*, Hochschultaschenbücher, vol. 112, Bibliographisches Institut, Mannheim, Wien, Zürich, 1966.

[2] Komaravolu Chandrasekharan, *Elliptic functions*, Grundlehren der math. Wiss., vol. 281, Springer, Berlin, Heidelberg, New York, Tokyo, 1985.

[3] Carl Gustav Jacob Jacobi, *Gesammelte Werke*, AMS Chelsea Publishing, Providence, 1969.

[4] Camille Jordan, *Cours d'Analyse*, 3rd ed., Vol. II, Jacques Gabay, Paris, 1991. réimpression autorisée de la 3. édition publiée par Gauthiers–Villars en 1913.

[5] David Mumford, *Tata Lectures on Theta*, Modern Birkhäuser Classics, Springer, 2007.

[6] Karl Weierstraß, *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen*, Göttingen, 1885. Nach Vorlesungen und Aufzeichnungen des Herrn K. Weierstrass bearbeitet und herausgegeben von H. A. Schwarz.

[7] André Weil, *Elliptic Functions according to Eisenstein and Kronecker*, Classics in Mathematics, Springer, Berlin, Heidelberg, New York, 1976, 1999.