GUIDE TO STOKES THEOREM

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Abstract. Guide to the theorem of Stokes.

Preface

In the winter term 1970/71 Günter Harder was lecturing on differential geometry for students of the 3rd semester (Infinitesimalrechnung 3) at University of Bonn. I was mentoring one of the groups doing the exercises. As the class started to become tough for the group I offered to run a workshop at the end of the term during the vacation. The material was determined by the students and I wrote it up during the next term. The draft was then given to the university printing office and photocopied and distributed in October 1971.

This is an English translation (and typeset in $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX) of the otherwise unchanged notes.

Mainz, April 27, 2002 B. E. Schwerdtfeger

Preface to the TeX Version 1.1. Corrected a misprint in the definition 1.4 of an atlas.

Mainz, February 25, 2005 B. E. Schwerdtfeger

Preface to the TeX Version 1.2. Corrected Mathematics Subject Classification.

Berlin, August 2, 2007 B. E. Schwerdtfeger

1. Manifolds

1.1. **Embedded manifolds.** Let E, F be *euclidean vector spaces* (finite dimensional vector spaces over $\mathbb{R}$).

Definition 1.1. A subset $M \subset E$ is called a C^r*–manifold* $(r \geq 1) :\Longleftrightarrow$
$\forall\, p \in M \quad \exists\, f : U \longrightarrow F \quad C^r$–differentiable with $p \in U \subset E$ open, such that

(1) $M \cap U = f^{-1}(0)$
(2) $Df(p) : E \longrightarrow F$ is surjective

Definition 1.2. The kernel in (2) is called *tangent space of M at the point p*:

$$T_pM = \operatorname{Ker} Df(p)$$

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Axiom (1) says: locally, M is the set of zeros of an r-times differentiable function.

Axiom (2) guaranties the existence of trivializing maps, also called *charts*, i.e. the following theorem holds (Theorem of implicit functions):

$\forall\ p \in M \quad \exists\ p \in U \subset E$ open and $\varphi : U \xrightarrow{\sim} V$ C^r–diffeomorphism onto an open set $V \subset E_1 \times E_2$ with the commutative diagram

$$\begin{tikzcd} \varphi : U \arrow[r, "\sim"] & V \\ \tilde{\varphi} : U \cap M \arrow[u, hook] \arrow[r, "\sim"] & V \cap E_1 \times \{0\} \arrow[u, hook] \end{tikzcd}$$

The induced map $\tilde{\varphi}$ is a *local chart.*

Example 1.1.

(1) open set $M = U \subset E$, $f : U \longrightarrow \{0\}$
(2) point $M = \{p\} \subset E$

$$\begin{aligned} f : E &\longrightarrow E \\ v &\longmapsto v - p \end{aligned}$$

(3) Graph Γ_f of a C^r–differentiable map

$$\begin{tikzcd} E \\ f : U \arrow[u, hook] \arrow[r] & F \end{tikzcd}$$

$M = \Gamma_f \subset U \times F \subset E \times F$, define

$$\begin{aligned} g : U \times F &\longrightarrow F \\ (u, v) &\longmapsto f(u) - v \end{aligned}$$

then the differential is surjective

$$\begin{aligned} Dg(u, v) : E \times F &\longrightarrow F \\ (s, t) &\longmapsto Df(u)s - t \end{aligned}$$

the tangent space is for $p \in M$, where $p = (u, f(u)), u \in U$

$$T_pM = T_p(\Gamma_f) = \Gamma_{Df(u)} \quad \text{graph of the differential}$$

(4) n–sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$

$$\begin{aligned} f : \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}, \qquad \mathbb{S}^n = f^{-1}(0) \\ v &\longmapsto \langle v, v \rangle - 1 \end{aligned}$$

the differential

$$\begin{aligned} Df(v) : \mathbb{R}^{n+1} &\longrightarrow \mathbb{R} \quad \text{is surjective for } v \neq 0 \\ w &\longmapsto 2\langle v, w \rangle \end{aligned}$$

the tangent space is for $p \in \mathbb{S}^n$

$$T_p\mathbb{S}^n = \{v \in \mathbb{R}^{n+1} \mid v \perp p\}$$

(5) Rotation surfaces in $\mathbb{R}^3$
  (a) $z = x^2 + y^2$ elliptical paraboloid
  (b) $x^2 \pm y^2 - z^2 = 1$ one- resp. two-branched hyperboloid

1.2. **Differentiable manifolds and morphisms.** Let M be a set, E a finite dimensional real vector space.

Definition 1.3 (chart)**.** A bijective map

$$x : U \xrightarrow{\sim} V \subset E$$

of a subset $U \subset M$ onto an open subset $V \subset E$ is called a *chart* (or also a *coordinate system*).

Notation: (U, x).

(U, x) is a chart around $p \quad :\Leftrightarrow \quad p \in U$.

Definition 1.4 (atlas)**.** A set of charts $A = \{x_\alpha : U_\alpha \longrightarrow E\}$ is a C^r*–atlas* for M $:\Longleftrightarrow$

(1) $M = \bigcup_\alpha U_\alpha$
(2) $\forall\ \alpha, \beta \quad x_\alpha(U_\alpha \cap U_\beta)$ is open
(3) $\forall\ \alpha, \beta \quad$ the map $x_{\beta\alpha}. = x_\beta \circ x_\alpha^{-1} | x_\alpha(U_\alpha \cap U_\beta)$

$$x_{\beta\alpha} : x_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\sim} x_\beta(U_\alpha \cap U_\beta) \qquad \text{is } C^r\text{–differentiable}$$

There exists exactly one *topology* on M for which the x_α are homeomorphisms: the U_α are declared *open* and they receive the topology via the charts x_α from $x_\alpha(U_\alpha)$. As the change of charts is a diffeomorphism, hence a homeomorphism, the topologies of U_α and U_β coincide on $U_\alpha \cap U_\beta$.

Caution : M does not need to be *separated* (Hausdorff) by these requirements !

Definition 1.5. Two atlas A, B are equivalent $\Longleftrightarrow$ $A \cup B$ is again an atlas.

An equivalence class D of C^r–atlas is called a C^r*–differentiable structure* of M.

The union of all atlas belonging to a structure D is the (uniquely determined) *maximal atlas*

$$A_{max} = \bigcup_{A \in D} A$$

A set M endowed with a differentiable structure is called a *differentiable manifold*. The charts in the maximal atlas are called admissible and are exclusively considered in what follows.

It is a fact that a topological manifold can have different differentiable structures. We are not going to elaborate on this.

Definition 1.6. A map of C^r–differentiable manifolds M, N

$$f : M \longrightarrow N$$

is called C^s–differentiable $(s \leq r)$ $:\Longleftrightarrow$
$\forall\ p \in M \quad \exists$ charts (U, x) around p and (V, y) around $f(p)$ such that

(1) $f(U) \subset V$
(2) $f_{yx} := y \circ f \circ x^{-1} : x(U) \longrightarrow y(V)$ is C^s–differentiable

Here is the situation in a diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N & \text{global} \\ U \arrow[u, hook] \arrow[r, "f|U"] \arrow[d, "x"'] & V \arrow[u, hook] \arrow[d, "y"] & \text{local} \\ x(U) \arrow[r, "f_{yx}"'] & y(V) & \end{tikzcd}$$

Remark. Read the comments to these definition in books on differential geometry, e.g. in LANG [5, pages 421–425].

The theme 'manifolds' lends itself to variations

- as *model spaces* E we can take Banach spaces (LANG [6])
- as base field we can take other fields than the reals $\mathbb{R}$, like the complex ($\mathbb{C}$) or p–adic numbers ($\mathbb{Q}_p$ or $\mathbb{C}_p$, the latter being the completion of the algebraic closure $\bar{\mathbb{Q}}_p$).

In this guide only *real C^∞–differentiable manifolds of finite dimension* are considered ($\dim M = \dim E$). The model space of a fixed manifold is fixed. In examples we take mostly $E = \mathbb{R}^n$. We also assume that the manifolds are *separated* and have a *countable basis*. In terms of the differentiable structure this signifies that the charts of an atlas should not "glue together too much" and they should not be "too many".

Example 1.2 (The n–sphere $\mathbb{S}^n$)**.** (see [3, page 2]).

$\mathbb{S}^n = \{p \in \mathbb{R}^{n+1} \mid |p| = 1\}$, the *north pole* is $p_N = (0, \dots, 0, 1)$, the south pole is $p_S = (0, \dots, 0, -1)$, the equator is $\mathbb{S}^n \cap \mathbb{R}^n \times \{0\} = \mathbb{S}^{n-1} \times \{0\}$, the equatorial hyper plane is $\mathbb{R}^n \times \{0\}$.

As charts we take the stereographic projection from the north resp. the south pole onto the euqatorial hyper plane ($\simeq \mathbb{R}^n$):

$x_N : \mathbb{S}^n - \{p_N\} \longrightarrow \mathbb{R}^n, \quad x_N(p) = \frac{(p_1,\dots,p_n)}{1-p_{n+1}}$

$x_S : \mathbb{S}^n - \{p_S\} \longrightarrow \mathbb{R}^n, \quad x_S(p) = \frac{(p_1,\dots,p_n)}{1+p_{n+1}}$

The change of charts is

$$\begin{aligned} x_{SN} = x_{NS} : \mathbb{R}^n - \{0\} &\longrightarrow \mathbb{R}^n - \{0\} \\ v &\longmapsto \frac{v}{|v|^2} \end{aligned}$$

which is C^∞–differentiable.

Example 1.3 (The n–dimensional projective space $\mathbb{P}^n$)**.** (see [3, page 20]).

The relation

$$b = \lambda \cdot a$$

for $a, b \in \mathbb{R}^{n+1} - \{0\}, \lambda \in \mathbb{R}^\times$ is an equivalence relation. The euqivalence class of an $a \in \mathbb{R}^{n+1} - \{0\}$ will be denoted $[a]$. The quotient of $\mathbb{R}^{n+1} - \{0\}$ modulo this relation is called the *real projective space*:

$$\mathbb{P}^n = \{[a] \mid a \in \mathbb{R}^{n+1} - \{0\}\}$$

The points $p \in \mathbb{P}^n$ correspond to the lines in $\mathbb{R}^{n+1}$; the point $p = [a] = \mathbb{R}^\times \cdot a$ corresponding to the line $\mathbb{R} \cdot a \subset \mathbb{R}^{n+1}$.

Let $U_i = \{[a] \mid a_i \neq 0\}$, $a = (a_0, \dots, a_n)$. Then we have $\mathbb{P}^n = U_0 \cup \dots \cup U_n$.

The map

$$\begin{aligned} x_i : U_i &\longrightarrow \mathbb{R}^n \\ [a] &\longmapsto \frac{1}{a_i} \cdot (a_0, \dots, \widehat{a_i}, \dots, a_n) \end{aligned}$$

is well defined: for $[a] = [b]$ we have $b = \lambda \cdot a$ and the factor λ cancels in the formula for the x_i.

The (U_i, x_i) constitute a C^∞–atlas.

Let's study the differentiable map

$$\begin{aligned} \pi : \mathbb{S}^n &\longrightarrow \mathbb{P}^n \\ a &\longmapsto [a] \end{aligned}$$

The fiber over a point $p \in \mathbb{P}^n$ is

$$\pi^{-1}(p) = \{a, -a\} \qquad \text{where } p = [a], |a| = 1$$

This gives a "visualization" of the projective space $\mathbb{P}^n$: diametrically opposed points on the sphere are glued together to give the projective space.

2. Tangent Bundle

2.1. **Tangent space T_pM.** Let M be a manifold with model space E.

For two charts

$$\begin{aligned} x : U &\longrightarrow E \\ y : V &\longrightarrow E \end{aligned}$$

the change of charts

$$y \circ x^{-1} : x(U \cap V) \longrightarrow y(U \cap V)$$

is a diffeomorphism of open sets in E, the derivative in any point must therefore be an automorphism of the vector space E:

$$D(y \circ x^{-1}) : x(U \cap V) \longrightarrow GL(E)$$

Definition 2.1. Define $\frac{\partial y}{\partial x}. = D(y \circ x^{-1}) \circ x$

$$\begin{aligned} \frac{\partial y}{\partial x} : U \cap V &\longrightarrow GL(E) \\ p &\longmapsto \frac{\partial y}{\partial x}|_p \end{aligned}$$

Let $C_p = \{x \mid x \text{ chart around } p\}$.

<u>Equivalence relation in $C_p \times E$</u>: $x, y \in C_p, v, w \in E$

$$(x, v) \sim (y, w) :\Longleftrightarrow \frac{\partial y}{\partial x}|_p v = w$$

Reflexivity: $\frac{\partial x}{\partial x}|_p = id$

Symmetry: $\frac{\partial x}{\partial y}|_p = \left(\frac{\partial y}{\partial x}|_p\right)^{-1}$

Transitivity: $\frac{\partial z}{\partial y}|_p \circ \frac{\partial y}{\partial x}|_p = \frac{\partial z}{\partial x}|_p \qquad$ chain rule

The equivalence class of (x, v) will be denoted by $(p, v)_x$ and is called *tangent vector*.

v is called a *representative* of the tangent vector in the chart x.

Definition 2.2.

$$T_pM = C_p \times E/\sim\, = \{(p, v)_x \mid x \text{ chart around } p,\ v \in E\}$$

Let $x \in C_p$, then to each tangent vector $t \in T_pM$ correpsonds exactly one representative $v \in E$ in the chart x:

Existence: $t = (p, w)_y$, let $v = \frac{\partial x}{\partial y}|_p w$, then $t = (p, v)_x$

Unicity: $(p, v)_x = (p, w)_x$, then $w = \frac{\partial x}{\partial x}|_p v = v$

Any chart $x \in C_p$ induces thus a canonical bijection

$$\begin{aligned} x_p : T_pM &\xrightarrow{\sim} E \\ t &\longmapsto v \qquad t = (p, v)_x \end{aligned}$$

<u>Vector space structure on T_pM</u>: The map x_p induces a vector space structure by transportation:

$$\begin{aligned} (p, v_1)_x + (p, v_2)_x &:= (p, v_1 + v_2)_x \\ \lambda \cdot (p, v)_x &:= (p, \lambda \cdot v)_x \end{aligned}$$

The vector space structure is independent from the chosen chart, as for another chart $y \in C_p$ the change of charts map $\frac{\partial y}{\partial x}|_p$ is $\mathbb{R}$–linear.

Definition 2.3 (Tangent bundle)**.** Let $U \subset M$ be open. Set $TU = \coprod_{p \in U} T_pM$ to be the disjoint union.

$$\begin{aligned} \tau_M = \tau : TM &\longrightarrow M \\ t &\longmapsto p \qquad \text{for } t \in T_pM \end{aligned}$$

The fiber over p is $\tau^{-1}(p) = T_pM$ and TM is the *tangential vector bundle* or *tangential bundle* for short.

2.2. **Manifold structure of the tangent bundle.** Any chart $x : U \longrightarrow E$ defines a bundle chart

$$\begin{aligned} \tau_x : TU &\longrightarrow E \times E \\ (p, v)_x &\longmapsto (x(p), v) \end{aligned}$$

Change of charts: $x : U \longrightarrow E$, $y : V \longrightarrow E$, then

$$\begin{aligned} \tau_y \circ \tau_x^{-1} : x(U \cap V) \times E &\longrightarrow y(U \cap V) \times E \\ (x(p), v) &\longmapsto \big(y(p), \frac{\partial y}{\partial x}|_p v\big) \end{aligned}$$

and this is differentiable.

TM is thus a manifold with model space $E \times E$, hence of dimension $\dim TM = 2 \cdot \dim M$.

The bundle projection τ is differentiable.

2.3. **Tangential mappings.** Let M, N be manifolds with model spaces E, F and let

$$f : M \longrightarrow N$$

be differentiable.

Let $p \in M$, $x \in C_p$, $y \in C_{f(p)}$,

$$x : U \longrightarrow E, \quad y : V \longrightarrow F, \quad \text{with } f(U) \subset V$$

Definition 2.4 (tangential map)**.** Define

$$\begin{aligned} T_p f : T_p M &\longrightarrow T_{f(p)} N \\ (p, v)_x &\longmapsto (f(p), D(f_{yx})(x(p))v)_y \end{aligned}$$

Here is the situation in a diagram

$$\begin{tikzcd} U \arrow[r, "f|U"] \arrow[d, "x"'] & V \arrow[d, "y"] \\ x(U) \arrow[r, "f_{yx}"'] & y(V) \end{tikzcd} \qquad \text{local}$$

The tangential map is defined on each fiber, therefore on the bundle, and this map

$$Tf : TM \longrightarrow TN$$

is differentiable, by inspecting the local situation:

$$\begin{tikzcd} TU \arrow[r, "Tf"] \arrow[d, "\tau_x"'] & TV \arrow[d, "\tau_y"] \\ x(U) \times E \arrow[r] & y(V) \times F \end{tikzcd}$$

where the lower map is

$$(x(p), v) \longmapsto \big(y \circ f(p), D(f_{yx})(x(p))v\big)$$

The following diagram is commutative

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\tau_M"'] & TN \arrow[d, "\tau_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

that is $\tau_N \circ Tf = f \circ \tau_M$.

<u>Chain rule for tangential maps</u>

$$M \longrightarrow N \longrightarrow P \qquad\qquad p \in M$$

We have $T_p(g \circ f) = T_{f(p)} g \circ T_p f : T_p M \longrightarrow T_{f(p)} N \longrightarrow T_{g \circ f(p)} P$ and $T(g \circ f) = Tg \circ Tf : TM \longrightarrow TN \longrightarrow TP$.

T is a functor from the category of manifolds into itself.

2.4. **Vector fields.**

Definition 2.5. A differentiable map
$$X : M \longrightarrow TM$$
with $\tau \circ X = id$ is called a (contravariant) *vector field* (or also tangential vector field).

Let the model space be $E = \mathbb{R}^m$, and $e_1, \ldots, e_m$ be the canonical basis, let $x = (x^1, \ldots, x^m) : U \longrightarrow \mathbb{R}^m$ be a local chart. Define a map
$$\frac{\partial}{\partial x^i} : U \longrightarrow TU$$
$$p \longmapsto \frac{\partial}{\partial x^i}|_p := (p, e_i)_x$$
Then the $\frac{\partial}{\partial x^1}|_p, \ldots \frac{\partial}{\partial x^m}|_p \in T_pM$ are a basis ($\forall\ p \in U$) and *locally* non-vanishing vector fields do exist.

3. Vector Bundles

3.1. **Categories, functors, natural transformations.** For the terminology of categories and functors see Lang, Algebra [7, chap. I, §11].

Let $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ be functors.

A family
$$t = (t_A)_{A \in \mathcal{A}} \in \prod_{A \in \mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(FA, GA)$$
is called a *natural transformation* from F to G $:\Longleftrightarrow$
$\forall\ f : A \longrightarrow A'$ morphisms in $\mathcal{A}$ we have commutative diagrams $Gf \circ t_A = t_{A'} \circ Ff$:
$$\begin{tikzcd} FA \arrow[r, "t_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FA' \arrow[r, "t_{A'}"'] & GA' \end{tikzcd}$$
Let $[\mathcal{A}, \mathcal{B}]$ be the category, the objects of which are the functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and the morphisms are defined by
$$\mathrm{Hom}_{[\mathcal{A},\mathcal{B}]}(F, G) \subset \prod_{A \in \mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(FA, GA)$$
the subset of natural transformations from F to G.

When $t : F \longrightarrow G$ and $u : G \longrightarrow H$ are given, then the composition $u \circ t$ will be the natural transformation $(u_A \circ t_A)_{A \in \mathcal{A}}$. Instead of 'natural transformation' it is thus natural to talk of *morphism of functors*.

Example 3.1. $[Open(M)^\circ, \mathcal{A}b]$ is the category of *abelian presheaves* on a topological space M (e.g. a manifold). In this case the category $Open(M)$ are the open sets of M with $\mathrm{Hom}(V, U) = \{V \hookrightarrow U\}$ if $V \subset U$, and $= \varnothing$ otherwise. A morphism of presheaves is therefore a family $t_U : F(U) \longrightarrow G(U)$ with
$$\begin{tikzcd} F(U) \arrow[r] \arrow[d] & G(U) \arrow[d] \\ F(V) \arrow[r] & G(V) \end{tikzcd}$$
commutative for $\forall\ V \subset U$.

3.2. **Multilinear algebra.** Let $\mathcal{V}$ be the category of finite dimensional $\mathbb{R}$ vector spaces,

$$\mathcal{V}(p,q) = \mathcal{V}^\circ \times \cdots \times \mathcal{V}^\circ \times \mathcal{V} \times \cdots \times \mathcal{V}$$

be the product of p–times $\mathcal{V}^\circ$ resp. q–times $\mathcal{V}$.

3.2.1. *Multilinear maps* $L^r : \mathcal{V}(r,1) \longrightarrow \mathcal{V}$. This functor is defined on the objects by:

$$L^r(V_1,\dots,V_r;W) = \{f : V_1 \times \cdots \times V_r \to W \quad \text{multilinear}\}$$

and on the morphisms by:

$$\begin{array}{cccc} \varphi_i : V_i' \to V_i, & \psi : W \to W' & & \\ L^r(\varphi_1,\dots,\varphi_r;\psi): & L^r(V_1,\dots,V_r;W) & \longrightarrow & L^r(V_1',\dots,V_r';W') \\ & \phi & \longmapsto & \psi \circ \phi \circ \varphi_1 \times \cdots \times \varphi_r \end{array}$$

Analogous definitions for

$$L^r, L^r_{alt}, L^r_{sym} : \mathcal{V}(1,1) \longrightarrow \mathcal{V}$$

where this time $L^r(V,W) = L^r(V,\dots,V;W)$.

We have $L^2(V_1,V_2;W) \simeq L^1(V_1, L^1(V_2,W))$ and it is easy to verify that this gives an isomorphism of functors

$$L^2 \simeq L^1 \circ id \times L^1 : \mathcal{V}(2,1) \to \mathcal{V}$$

All isomorphisms in the class [4] are isomorphisms of functors: this is exactly the meaning of *canonical*.

3.2.2. *Tensor functor* $\otimes : \mathcal{V}(0,2) \longrightarrow \mathcal{V}$.

Theorem 3.1. *There is a functor* $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ *and to any two vector spaces* $V_1, V_2 \in \mathcal{V}$ *a bilinear map* $\sigma : V_1 \times V_2 \longrightarrow V_1 \otimes V_2$ *such that*

$$\begin{array}{ccc} \mathrm{Hom}(V_1 \otimes V_2, W) & \xrightarrow{\sim} & L^2(V_1,V_2;W) \\ \varphi & \longmapsto & \varphi \circ \sigma \end{array}$$

Proof. Let $V_{12} = \mathbb{R}\langle V_1 \times V_2 \rangle$ be the vector space with basis (v_1,v_2) where $v_1 \in V_1, v_2 \in V_2$.

Let $V_0 \subset V_{12}$ be the subspace generated by all vectors of the form

$$\begin{aligned} (v_1+v_1',v_2) - (v_1,v_2) - (v_1',v_2) \\ (v_1,v_2+v_2') - (v_1,v_2) - (v_1,v_2') \\ \lambda \cdot (v_1,v_2) - (\lambda v_1, v_2) \\ \lambda \cdot (v_1,v_2) - (v_1, \lambda v_2) \end{aligned} \tag{1}$$

Define the tensor product as $V_1 \otimes V_2 = V_{12}/V_0$ and the map σ by

$$\begin{array}{cccccc} \sigma : & V_1 \times V_2 & \longrightarrow & V_1 \otimes V_2 & \\ & (v_1,v_2) & \longmapsto & v_1 \otimes v_2 & = (v_1,v_2) + V_0 \end{array}$$

Because of the relations (1) the map σ is bilinear.

The isomorphism $\mathrm{Hom}(V_1 \otimes V_2, W) \xrightarrow{\sim} L^2(V_1,V_2;W)$ is then clear. □

Exercise 3.1. Make explicit that this is functorial in V_1, V_2, W.

Remark. $\otimes$ will be defined on the morphisms as follows: $\varphi_i : V_i \to V_i' \quad (i = 1, 2)$

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "\sigma"] \arrow[d, "\varphi_1 \times \varphi_2"'] & V_1 \otimes V_2 \arrow[d, dashed, "\varphi"] \\ V_1' \times V_2' \arrow[r, "\sigma'"'] & V_1' \otimes V_2' \end{tikzcd}$$

In a similar way one defines

$$T^p_q : \mathcal{V}(q,p) \longrightarrow \mathcal{V}$$

$$T^p_q(W_1, \dots, W_q, V_1, \dots, V_p) = V_1 \otimes \dots \otimes V_p \otimes W_1^* \otimes \dots \otimes W_q^*$$

For $V_1 = \dots = V_p = W_1 = \dots = W_q = E$ one defines again $T^p_q E = E^{\otimes p} \otimes E^{*\otimes q}$, the elements of this vector space are called *(mixed) tensors of type* (p,q) or also p–fold contravariant, q–fold covariant.

Remark. Be not confused by the terminolgy: the functor T^p_q is p–fold covariant (in the variables $V_1, \dots, V_p$) and q–fold contravariant (in the variables $W_1, \dots, W_q$). For the tensors itself (the elements of $E^{\otimes p} \otimes E^{*\otimes q}$) the terminology was introduced fifty years earlier (opposite to what appears natural today).

3.2.3. *r–fold exterior product $\bigwedge^r : \mathcal{V} \longrightarrow \mathcal{V}$.*

Theorem 3.2. *There is a functor $\bigwedge^r : \mathcal{V} \longrightarrow \mathcal{V}$ and to any vector spaces $V \in \mathcal{V}$ an r–fold multilinear alternating map $\tau : V^r \longrightarrow \bigwedge^r V$ such that*

$$\begin{array}{ccc} \mathrm{Hom}(\bigwedge^r V, W) & \xrightarrow{\sim} & L^r_{alt}(V,W) \\ \varphi & \longmapsto & \varphi \circ \tau \end{array}$$

Proof. Let $V_1 = T^r_0 = V \otimes \dots \otimes V \quad r$–times. Let $V_0 \subset V_1$ be the subspace genrated by the vectors of the form $v_1 \otimes \dots v_r$ with $v_i = v_j$ for at least one index pair $i \neq j$.

Define $\bigwedge^r V = V_1/V_0$ and denote the image of $v_1 \otimes \dots \otimes v_r$ in the quotient by $v_1 \wedge \dots \wedge v_r$, the map τ is defined by

$$\begin{array}{rccccl} \tau : & V^r & \longrightarrow & \bigwedge^r V & & \\ & (v_1, \dots, v_r) & \longmapsto & v_1 \wedge \dots \wedge v_r & = v_1 \otimes \dots \otimes v_r + V_0 \end{array}$$

To a map $\varphi : V \to V'$ the map $\bigwedge^r \varphi : \bigwedge^r V \longrightarrow \bigwedge^r V'$ is given by
$\bigwedge^r \varphi(v_1 \wedge \dots \wedge v_r) = \varphi(v_1) \wedge \dots \wedge \varphi(v_r)$ □

3.2.4. *Exterior Algebra $\bigwedge^\bullet : \mathcal{V} \longrightarrow Alg$.* Define the exterior algebra functor by $\bigwedge^\bullet(V) = \bigwedge^0 V \oplus \bigwedge^1 V \oplus \bigwedge^2 V \oplus \dots$

Theorem 3.3. *If $v_1, \dots, v_n$ is a basis of V, then the $v_{i_1} \wedge \dots \wedge v_{i_r}$ with $1 \leq i_1 < \dots < i_r \leq n$ is a basis of $\bigwedge^r V$ (and thus $\dim \bigwedge^r V = \binom{n}{r}$).*

Proof. See Lang Algebra [7, XIX, §1, prop. 1.1] □

3.2.5. *A bilinear form.* Let E be a finite dimensional vector space and consider

$$\begin{array}{ccc} \Phi : E^{*r} \times E^r & \longrightarrow & \mathbb{R} \\ (\lambda_1, \dots, \lambda_r, v_1, \dots, v_r) & \longmapsto & \det(\lambda_i(v_j)) \end{array}$$

Φ is multilinear and vanishes if either $\lambda_i = \lambda_j$ or $v_i = v_j$ for $i \neq j$. Hence this induces a bilinear map

$$\begin{array}{ccc} \langle\ ,\ \rangle : \bigwedge^r E^* \oplus \bigwedge^r E & \longrightarrow & \mathbb{R} \\ \langle \lambda_1 \wedge \dots \wedge \lambda_r, v_1 \wedge \dots \wedge v_r \rangle & = & \det(\lambda_i(v_j)) \end{array}$$

3.3. **Construction of vector bundles on manifolds.**

Definition 3.1. A differentiable map

$$\pi : V \longrightarrow M$$

is called a *vector bundle* on M with fiber F $(F \in \mathcal{V}) \Longleftrightarrow$

(1) There is a vector space structure on each fiber

$$\pi^{-1}(p) \in \mathcal{V}$$

(2) local triviality: each point of M has an open neighbourhood U and a diffeomorphism

$$\begin{tikzcd} h : \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

such that $\forall\, p \in U \quad \pi^{-1}(p) \xrightarrow{\sim} \{p\} \times F$ is a vector space isomorphism.

Let $\pi : V \to M$ be a vector bundle, $U \subset M$ open. By $\pi|U$ is denoted the vector bundle $\pi^{-1}(U) \to U$ on U.

$$\Gamma(\pi) = \{\sigma : M \to V \mid \text{differentiable such that } \pi \circ \sigma = 1_M\}$$

is the real vector space of differentiable *sections* of the vector bundle. This vector space is in general no longer finite dimensional.

3.3.1. *Tensor bundles.* One obtains plenty of vector bundles on a manifold by applying the functors of multilinear algebra to the fibers of the tangential bundle $\tau : TM \to M$. Thus, we are carrying out the remark in LANG's Algebra about differential geometry, [7, XVI, §5] (*tensors of type* L, where L is a functor on vector spaces, see [7, p. 628]). τ itself is an example of a vector bundle, the fiber is E. The functor T^p_q gives us

$$\tau^p_q : T^p_q M \longrightarrow M$$

the vector bundle of p–fold contravariant, q–fold covariant tensors. The sections are called *tensor fields of type* (p, q).

For type $(1, 0)$ they are called *vector fields*, for type $(0, 1)$ they are the *Pfaffian forms* or *differential forms of degree* 1.

The functor $\bigwedge^r$ leads to the bundle $\pi^r : \bigwedge^r TM \to M$, the alternating contravariant tensors. Similar the dual bundle $\omega^r : \bigwedge^r T^*M \to M$ of alternating covariant tensors. Its construction will be carried thru in detail, as its sections (the *differential forms*) are studied further on.

3.3.2. $\omega^r_M : \bigwedge^r T^*M \longrightarrow M$.

Definition 3.2.

$$T^*_p M = (T_p M)^* \qquad \bigwedge^r T^*U = \bigcup_{p \in U} \bigwedge^r T^*_p M \quad U \subset M \text{ open}$$

$\omega^r : \bigwedge^r T^*M \longrightarrow M$ is the projection, mapping $\bigwedge^r T^*_p M$ onto p.

3.3.3. *Manifold structure on $\bigwedge^r T^*M$.* To a chart $x : U \to E$ of M we construct the associated bundle chart

$$\bar{x} : \bigwedge^r T^*U \xrightarrow{\sim} x(U) \times \bigwedge^r E^* \subset E \times \bigwedge^r E^*$$

By $x_p : T_pM \xrightarrow{\sim} E$ for $p \in U$ we get

$$\bigwedge^r (x_p^{-1})^* : \bigwedge^r T_p^*M \longrightarrow \bigwedge^r E^*$$

and $\bar{x}$ maps the fiber to $\{x(p)\} \times \bigwedge^r E^*$.

Change of charts: let $y : V \to E$ be another chart, then

$$\begin{array}{rccc} \bar{y} \circ \bar{x}^{-1} : & x(U \cap V) \times \bigwedge^r E^* & \longrightarrow & y(U \cap V) \times \bigwedge^r E^* \\ & (x(p), \alpha) & \longmapsto & (y(p), \bigwedge^r (y_p^{-1})^* \circ \bigwedge^r (x_p)^* \alpha) \end{array}$$

Now we have $\bigwedge^r (y_p^{-1})^* \circ \bigwedge^r (x_p)^* = \bigwedge^r (x_p \circ y_p^{-1})^* = \bigwedge^r (\frac{\partial x}{\partial y}|_p)^*$, as $x_p \circ y_p^{-1} = \frac{\partial x}{\partial y}|_p$. Hence

$$\begin{array}{rccc} \bar{y} \circ \bar{x}^{-1} : & x(U \cap V) \times \bigwedge^r E^* & \longrightarrow & y(U \cap V) \times \bigwedge^r E^* \\ & (x(p), \alpha) & \longmapsto & (y(p), \bigwedge^r (\frac{\partial x}{\partial y}|_p)^* \alpha) \end{array}$$

is differentiable.

Remark. $\bigwedge^0 T^*M = M \times \mathbb{R}$ is the trivial line bundle (a vector bundle is a *line bundle*, if the fiber dimension is 1)

The previuosly defined bilinear form (defined on the fibers) induces

$$\langle\ ,\ \rangle : \bigwedge^r T^*M \oplus \bigwedge^r TM \longrightarrow M \times \mathbb{R}$$

4. Differential Forms

4.1. **Sheaves.**

Definition 4.1. A presheaf of real vector spaces $F : Open(M)^\circ \longrightarrow \mathcal{V}$ on a topological space M is a sheaf $:\Longleftrightarrow$

$\forall\, U \subset M$ open

$\forall\, U = \bigcup_{\alpha \in I} U_\alpha$ open coverings

$\forall\, (s_\alpha)_\alpha$ families of sections $s_\alpha \in F(U_\alpha)$ with the property $s_\alpha | U_\alpha \cap U_\beta = s_\beta | U_\alpha \cap U_\beta$

$\exists! s \in F(U)$ with $s | U_\alpha = s_\alpha$

This is called the *glueing property of sheaves* and the condition $s_\alpha | U_\alpha \cap U_\beta = s_\beta | U_\alpha \cap U_\beta$ is the *glueing condition*

Intuitively this signifies: You only need to define a $s \in F(U)$ locally on the U_α, $s_\alpha \in F(U_\alpha)$, and look at it to satisfy the glueing condition – the family $(s_\alpha)_\alpha$ is glueing together to a unique s automatically.

To make this clearer, let us be given $U = \bigcup_\alpha U_\alpha$ and mappings $f_\alpha : U_\alpha \longrightarrow \mathbb{R}$. Assume that $f_\alpha | U_\alpha \cap U_\beta = f_\beta | U_\alpha \cap U_\beta$. Define

$$f : U \longrightarrow \mathbb{R}$$

by $f(p) = f_\alpha(p)$ if $p \in U_\alpha$. Because of the glueing condition this is well defined.

As continuity (differentiability) is a *local* property, the glued map f is continuous (differentiable) if the f_α are.

From this follows at once: if $\pi : V \to M$ is a vector bundle, then $U \mapsto \Gamma(\pi | U)$ is a sheaf: the *sheaf of differentiable sections*.

4.1.1. *Sheaf of differential forms.* The sections of the bundle $\omega^r_M : \bigwedge^r T^*M \longrightarrow M$ are called *differential forms of degree* r (r–forms).

$$\begin{array}{rccc} \Omega^r : & Open(M)^\circ & \longrightarrow & \mathcal{V} \\ & U & \longmapsto & \Gamma(\omega^r|U) \end{array}$$

denotes the *sheaf of* r*–forms*. 1–forms are also called *Pfaffian forms*.

Because of the triviality of ω^0 the 0–forms are identified with the functions $M \to \mathbb{R}$.

$\mathcal{C}^\infty = \Omega^0$ is the *sheaf of rings of structure functions* – or short the *structure sheaf* – of M.

Definition 4.2. For $\omega \in \Omega^r(M), \eta \in \Omega^s(M)$ we define $\omega \wedge \eta \in \Omega^{r+s}(M)$ by $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$.

4.2. **The differential.** Let $f : M \to \mathbb{R}$ be given, the *differential of* f is defined by

$$\begin{array}{rccc} df : & M & \longrightarrow & T^*M \\ & p & \longmapsto & df_p \end{array}$$

where $df_p : T_pM \to \mathbb{R}$ is the composition of $T_pf : T_pM \to T_{f(p)}\mathbb{R}$ and $T_{f(p)}\mathbb{R} \xrightarrow{\sim} \mathbb{R}$ (the chart $id : \mathbb{R} \xrightarrow{\sim} \mathbb{R}$ on $\mathbb{R}$), hence

$$\begin{array}{rccc} df_p : & T_pM & \longrightarrow & \mathbb{R} \\ & (p,v)_x & \longmapsto & D(f \circ x^{-1})|_{x(p)}(v) \end{array}$$

4.2.1. *Partial derivative.* Let the model space be $E = \mathbb{R}^n$. Let $x : U \longrightarrow \mathbb{R}^n$ be a chart and $f : U \longrightarrow \mathbb{R}$ be a function. We define

$$\frac{\partial f}{\partial x^i} := \langle df, \frac{\partial}{\partial x^i} \rangle$$

These we get functions:

$$\begin{array}{rccc} \frac{\partial f}{\partial x^i} : & U & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & D(f \circ x^{-1})|_{x(p)} e_i \end{array}$$

Analogously we define

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial (\frac{\partial f}{\partial x^j})}{\partial x^i}$$

etc.

4.2.2. *Relations.*

(1) $df = \sum_i \frac{\partial f}{\partial x^i} \cdot dx^i$
(2) $\frac{\partial}{\partial y^j} = \sum_i \frac{\partial x^i}{\partial y^j} \cdot \frac{\partial}{\partial x^i}$
(3) $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$
(4) $\langle dx^i, \frac{\partial}{\partial x^j} \rangle = \delta^i_j$

in particular is $dx^1_p, \ldots, dx^n_p \in T^*_pM$ the dual basis to $\frac{\partial}{\partial x^1}|_p, \ldots \frac{\partial}{\partial x^n}|_p \in T_pM$

Proof. The first two relations follow from the last one, which one is obvious. As to the symmetry of the second derivative, it follows from

$$\frac{\partial^2 f}{\partial x^i \partial x^j}|_p = \langle d(\frac{\partial f}{\partial x^j})_p, \frac{\partial}{\partial x^i}|_p \rangle = D(\frac{\partial f}{\partial x^j} \circ x^{-1})|_{x(p)} e_i = D^2(f \circ x^{-1})|_{x(p)}(e_i, e_j)$$

and this is symmetric. $\square$

4.2.3. *Local representation of differential forms.* Let $\omega : M \longrightarrow \bigwedge^r T^*M$ be an r–form, $x : U \to \mathbb{R}^n$ a chart, then

$$\omega|U = \sum_{i_1<\dots<i_r} f_{i_1\dots i_r} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

where

$$f_{i_1\dots i_r} = \langle \omega|U, \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_r}} \rangle : U \longrightarrow \mathbb{R} \quad \text{differentiable}$$

Proof. $\omega_p \in \bigwedge^r T^*_pM$ can be uniquely expressed by the basis $dx^{i_1}_p \wedge \dots \wedge dx^{i_r}_p$, with coefficients $f_{i_1\dots i_r}(p) \in \mathbb{R}$. The differentiability of the $f_{\dots}$ follows from the relations above.

Remark. Use the fact that

$$\frac{\partial}{\partial x^{i_1}}|_p \wedge \dots \wedge \frac{\partial}{\partial x^{i_r}}|_p \in \bigwedge^r T_pM$$

is the dual basis to the

$$dx^{i_1}_p \wedge \dots \wedge dx^{i_r}_p \in \bigwedge^r T^*_pM$$

□

4.3. **Calculus of differential forms.**

Theorem 4.1. *There is a family of $\mathbb{R}$–linear maps*

$$d^r_U : \Omega^r(U) \longrightarrow \Omega^{r+1}(U) \quad \textit{exterior derivation}$$

such that

1. $d^0f = df$ *is the differential for* $f \in \mathcal{C}^\infty(U)$
2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$ *for* $\omega \in \Omega^r(U), \eta \in \Omega^s(U)$
3. $d^{r+1} \circ d^r = 0$
4. $d^r : \Omega^r \longrightarrow \Omega^{r+1}$ *is a morphism of sheaves*

Proof. As to the uniqueness: because of the last property and as the Ω^r are sheaves it suffices to consider those U with charts $x : U \to \mathbb{R}^n$. $\omega \in \Omega^r(U)$ is of the form $\omega|U = \sum f_{i_1\dots i_r} \cdot dx^{i_1} \wedge \dots \wedge dx^{i_r}$ and the first relations imply then that

$$d\omega|U = \sum df_{i_1\dots i_r} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r} \tag{$*$}$$

Now to existence: we necessarily define the exterior derivation locally by the formula ($*$). We show independence of the chosen chart:

$$dx^i = \sum_j \frac{\partial x^i}{\partial y^j} \cdot dy^j$$

hence

$$\omega = \sum f_{i_1\dots i_r} \cdot \frac{\partial x^{i_1}}{\partial y^{j_1}} \dots \frac{\partial x^{i_r}}{\partial y^{j_r}} dy^{j_1} \wedge \dots \wedge dy^{j_r}$$

Now the terms

$$d\frac{\partial x^i}{\partial y^\beta} \wedge dy^{j_1} \wedge \dots \wedge dy^{j_r} = \sum_\alpha \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} dy^\alpha \wedge \dots \wedge dy^\beta \wedge \dots$$

vanish, as they occur twice with opposite signs.

Hence we get

$$d\omega = \sum \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{j_r}} df_{i_1 \dots i_r} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_r}$$

The relation (1) is satisfied by definition. (2) holds by $d(f \cdot g) = g \cdot df + f \cdot dg$. (3) holds by $ddf = \sum_j d\frac{\partial f}{\partial x^j} dx^j = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0$. The fact (4) is trivial (the construction being compatible with restriction to smaller open subsets). $\square$

4.4. **Complex differential forms.** [8]

Let $U \subset \mathbb{R}^m$ be open, $x^1, \dots, x^m : U \to \mathbb{R}$ and $L_{\mathbb{R}}(\mathbb{R}^m, \mathbb{C})$ the space of complex valued $\mathbb{R}$–linear maps $\mathbb{R}^m \to \mathbb{C}$, this is complex vector space of dimension m. We consider the exterior algebra over $\mathbb{C}$ (!)

$$\textstyle\bigwedge_{\mathbb{C}}^r L_{\mathbb{R}}(\mathbb{R}^m, \mathbb{C}) \qquad 0 \le r \le m.$$

We define the *complex differential forms* by

$$\mathcal{E}^r(U) = \{\omega : U \to \textstyle\bigwedge_{\mathbb{C}}^r L_{\mathbb{R}}(\mathbb{R}^m, \mathbb{C}) \mid \text{differentiable maps}\}$$

Let $f : U \to \mathbb{C}$, $f = u + i \cdot v$, then $df = du + i \cdot dv : U \to L_{\mathbb{R}}(\mathbb{R}^m, \mathbb{C})$, where $du, dv : U \to L_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}) \subset L_{\mathbb{R}}(\mathbb{R}^m, \mathbb{C})$.

$\mathcal{E}^r(U)$ is free over $\mathcal{E}^0(U)$ with basis $dx^{i_1} \wedge \cdots \wedge dx^{i_r}$, $i_1 < \cdots < i_r$.

Now let $m = 2n$ be even: $U \subset \mathbb{R}^{2n} = \mathbb{C}^n$. We are going to rename the coordinates:

$$x_1, y_1, \dots, x_n, y_n : U \longrightarrow \mathbb{C}$$

with $x_\alpha = x^{2\alpha-1}, y_\alpha = x^{2\alpha}$ and we introduce the *complex* coordinates

$$z_\alpha = x_\alpha + iy_\alpha : U \longrightarrow \mathbb{C}$$

$$\bar{z}_\alpha = x_\alpha - iy_\alpha : U \longrightarrow \mathbb{C}$$

Let $\mathcal{E}^{p,q}(U) \subset \mathcal{E}^r(U)$ be the subspace generated by the

$$dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_p} \wedge d\bar{z}_{\beta_1} \wedge \cdots \wedge d\bar{z}_{\beta_q}$$

for the sets of indices

$$\begin{gathered} 1 \le \alpha_1 < \cdots < \alpha_p \le n \quad 0 \le p \le n \\ 1 \le \beta_1 < \cdots < \beta_q \le n \quad 0 \le q \le n \\ \text{with} \quad p + q = r;\ 0 \le r \le 2n. \end{gathered}$$

The sheaves $\mathcal{E}^{p,q}$ are locally free over $\mathcal{E}^0$ of rank $\binom{n}{p} \cdot \binom{n}{q}$, and $\mathcal{E}^r = \bigoplus_{p+q=r} \mathcal{E}^{p,q}$.

One defines complex vector fields, interpreted as differential operators

$$\frac{\partial}{\partial z_\alpha} = \frac{1}{2}\Big(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\Big),\ \frac{\partial}{\partial \bar{z}_\alpha} = \frac{1}{2}\Big(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\Big) : \mathcal{E}^0 \longrightarrow \mathcal{E}^0$$

and for $f \in \mathcal{E}^0(U)$

$$\partial f := \sum_\alpha \frac{\partial f}{\partial z_\alpha} dz_\alpha \in \mathcal{E}^{1,0}(U)$$

$$\bar{\partial} f := \sum_\alpha \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \in \mathcal{E}^{0,1}(U)$$

We then have $d = \partial + \bar{\partial} : \mathcal{E}^0 \longrightarrow \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1} = \mathcal{E}^1$.

We have analogous definitions for the differential forms:

$$\begin{aligned}
\omega \in \mathcal{E}^{p,q}(U), \qquad \omega &= \sum f_{\alpha_1 \ldots \beta_q} dz_{\alpha_1} \wedge \cdots \wedge d\bar{z}_{\beta_q} \\
\partial\omega &= \sum \partial f_{\alpha_1 \ldots \beta_q} \wedge dz_{\alpha_1} \wedge \cdots \wedge d\bar{z}_{\beta_q} \\
\bar{\partial}\omega &= \sum \bar{\partial} f_{\alpha_1 \ldots \beta_q} \wedge dz_{\alpha_1} \wedge \cdots \wedge d\bar{z}_{\beta_q}
\end{aligned}$$

This gives us maps

$$\begin{aligned}
\partial &: \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p+1,q} \\
\bar{\partial} &: \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q+1}
\end{aligned}$$

and we have $d = \partial + \bar{\partial}$. As $d \circ d = 0$ this gives $\partial \circ \partial = 0,\ \bar{\partial} \circ \bar{\partial} = 0,\ \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0.$

Let $\mathcal{H}^p(U) = \{\omega \in \mathcal{E}^{p,0}(U) \mid \bar{\partial}\omega = 0\}$ is the space of *holomorphic differential forms*. We have an exact diagram of sheaves (Dolbeault–Serre):

$$\begin{tikzcd}
 & 0 & 0 & 0 & & 0 & \\
0 \arrow[r] & \mathcal{H}^n \arrow[r] \arrow[u] & \mathcal{E}^{n,0} \arrow[r, "\bar{\partial}"] \arrow[u] & \mathcal{E}^{n,1} \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{E}^{n,n} \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] & & \vdots \arrow[u] & \\
0 \arrow[r] & \mathcal{H}^1 \arrow[r] \arrow[u] & \mathcal{E}^{1,0} \arrow[r, "\bar{\partial}"] \arrow[u] & \mathcal{E}^{1,1} \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{E}^{1,n} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{H}^0 \arrow[r] \arrow[u] & \mathcal{E}^{0,0} \arrow[r, "\bar{\partial}"] \arrow[u, "\partial"] & \mathcal{E}^{0,1} \arrow[r] \arrow[u, "\partial"] & \cdots \arrow[r] & \mathcal{E}^{0,n} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[u] & \bar{\mathcal{H}}^0 \arrow[r] \arrow[u] & \bar{\mathcal{H}}^1 \arrow[r] \arrow[u] & \cdots \arrow[r] & \bar{\mathcal{H}}^n \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

5. Stokes' Theorem

5.1. Partitions of 1.

Definition 5.1 (locally finite)**.** A family $(U_\alpha)_{\alpha \in I}$ of subsets of a topological space M is called *locally finite* $\Longleftrightarrow \forall\, p \in M, \quad \exists$ open neighbourhood V of p which meets only a finite subset of the U_α: for almost all $\alpha \in I$ we have $U_\alpha \cap V = \varnothing$.

Definition 5.2 (refinement)**.** Let $(U_\alpha)_{\alpha \in I}, (V_\beta)_{\beta \in J}$ be coverings of M.

$(V_\beta)_{\beta \in J}$ is a *refinement* of $(U_\alpha)_{\alpha \in I} \quad \Longleftrightarrow \exists\ \lambda : J \to I$ such that $V_\beta \subset U_{\lambda(\beta)}$ $\forall\, \beta \in J$. ("Each V_β is contained in a certain U_α").

Definition 5.3 (partition of 1)**.** Let M be a manifold, $(U_\alpha)_{\alpha \in I}$ an open covering of M.

A family $(\varphi_\alpha)_{\alpha \in I}$ of differentiable maps

$$\varphi_\alpha : M \longrightarrow \mathbb{R}$$

is called an *associated partition of 1* to $(U_\alpha)_{\alpha \in I} \quad \Longleftrightarrow$

(1) $\varphi_\alpha(p) \geq 0 \quad \forall\, \alpha \in I, p \in M$

(2) $\operatorname{supp}\varphi_\alpha \subset U_\alpha$
(3) $(\operatorname{supp}\varphi_\alpha)_{\alpha\in I}$ is locally finite
(4) $\sum_{\alpha\in I}\varphi_\alpha(p) = 1 \quad \forall\, p \in M$ (the sum is finite because of (3)).

Theorem 5.1. *To each open covering of a manifold there exists an associated partition of* 1*.*

Remark. In the construction of the partition of 1 the assumptions made in section 1.2 on the topology of M – i.e. to be *separated* and having a *countable basis* – are crucial.

Proofs are to be found in [3, §8.3], [6, chap. II,§3], [1, XVI,4], [2, III.2.12].

We will first derive two Lemmas.

Lemma 5.2. *Let $(U_\alpha)_{\alpha\in I}$ be an open covering of M. Then there exists an atlas*

$$x_\beta : V_\beta \longrightarrow \mathbb{R}^n \qquad \beta \in J$$

(and we can even take J to be countable*) such that*

(1) *$(V_\beta)_{\beta\in J}$ is a locally finite refinement of $(U_\alpha)_{\alpha\in I}$*
(2) $x_\beta(V_\beta) = K(0,3) = \{x \in \mathbb{R}^n \mid \|x\| < 3\}$
(3) *$(W_\beta)_{\beta\in J}$ is also a covering of M, where $W_\beta = x_\beta^{-1}(K(0,1)) \subset V_\beta$*

Proof. Let $(B_n)_{n\in\mathbb{N}}$ be a basis of the topology of M with $\overline{B_n}$ compact. Exhaust M by an increasing sequence of compact sets K_n such that K_n is contained in the interior of K_{n+1}:

$$M = \bigcup_{n\in\mathbb{N}} K_n\,, \qquad K_n \subset \overset{\circ}{K}_n$$

Construction of these K_n by induction on n.

$$K_1 = \overline{B_1}$$

Let K_n already be constructed with $B_n \subset K_n$. K_n is covered by finitely many of the B_i, say by $K_n \subset \bigcup_{i=1}^k B_i$. Put

$$K_{n+1} = \overline{B}_{n+1} \cup \bigcup_{i=1}^{k} \overline{B}_i$$

With the K_n being constructed, we next consider $K_{n+1} - \overset{\circ}{K}_n \subset \overset{\circ}{K}_{n+2} - K_{n-1}$, the smaller set being compact, the latter open (put $K_n = \varnothing$ for $n \leq 0$). To any $p \in K_{n+1} - \overset{\circ}{K}_n$ there is an $\alpha \in I$ such that $p \in U_\alpha$ and a chart $x : V_p \xrightarrow{\sim} K(0,3)$ around p such that

$$p \in x^{-1}(K(0,1)) \subset V_p \subset (\overset{\circ}{K}_{n+2} - K_{n-1}) \cap U_\alpha$$

Finitely many of the V_p cover the $K_{n+1} - \overset{\circ}{K}_n$. We therefore get charts

$$x_{ni} : V_{ni} \xrightarrow{\sim} K(0,3) \qquad i = 1,\dots,r_n$$

with $W_{ni} = x_{ni}^{-1}(K(0,1))$ and

$$K_{n+1} - \overset{\circ}{K}_n \subset \bigcup_{i=1}^{r_n} W_{ni} \subset \bigcup_{i=1}^{r_n} V_{ni} \subset \overset{\circ}{K}_{n+2} - K_{n-1}$$

and each V_{ni} lies in at least one U_α. As each of the V_{ni} meets only finitly many of the V_{mj} (there are only a finite number of them in $\overset{\circ}{K}_{n+2} - K_{n-1}$), the (countable) family V_{mj} is the searched for locally finite refinement of $(U_\alpha)_{\alpha\in I}$. $\square$

Lemma 5.3. *There exists a differentiable map*

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$$

such that

$$\begin{aligned} \psi(x) &= 1 \qquad \textit{for } \|x\| < 1 \\ \psi(x) &= 0 \qquad \textit{for } \|x\| \geq 2 \end{aligned}$$

Proof. We proceed in three steps.

(1) $\lim_{t\to 0, t>0} e^{-1/t} \cdot t^{-n} = 0$ for all $n \in \mathbb{N}$, as we have $e^{1/t} \geq \frac{t^{-n-1}}{(n+1)!}$ by the exponential series, which implies $0 \leq e^{-1/t} \cdot t^{-n} \leq (n+1)! \cdot t$.
(2) Let $f : (0, \infty) \to \mathbb{R}$ be the function $f(t) = e^{-1/t}$, then we have
  (a) $D^n f(t) = f(t) \cdot t^{-2n} \cdot P_n(t)$ where $P_n \in \mathbb{R}[X]$ is a polynomial of degree $\deg P_n \leq n - 1$ with the recursion formula
   - $P_1 = 1$
   - $P_{n+1} = (1 - 2nX)P_n + X^2 \cdot P_n'$

   (calculate by induction $n \to n + 1$).
  (b) $\lim_{t\to 0, t>0} D^n f(t) = 0$ (follows immediately from the previous results)
(3) Let finally

$$\begin{aligned} &h : \mathbb{R} \to \mathbb{R} \\ &h(t) = \exp(\frac{-1}{t-a}) \cdot \exp(\frac{-1}{b-t}) \qquad a < t < b \\ &h(t) = 0 \qquad\qquad \text{otherwise} \end{aligned}$$

This is C^∞–differentiable by (2). Hence also the function $H : \mathbb{R} \to \mathbb{R}$ defined by $H(t) = \int_t^b h$ is C^∞–differentiable, which has the property $H(t) = \int_a^b h$ constant for $t \leq a$ and $H(t) = 0$ for $t \geq b$. We can now construct $g : \mathbb{R} \to \mathbb{R}$ C^∞–differentiable, with $g(t) = 1$ for $|t| \leq 1$ and $g(t) = 0$ for $|t| \geq 4$.

Finally we choose for ψ: $\psi(x) = g(\|x\|^2)$, which is C^∞–differentiable by the chain rule and we are done.

□

Proof of theorem 5.1. Let $(U_\alpha)_{\alpha\in I}$ be the given open covering, let $x_\beta : V_\beta \to \mathbb{R}^n$ be the atlas of lemma 5.2, let $\lambda : J \to I$ be the refinement mapping $V_\beta \subset U_{\lambda(\beta)}$. Put $T_\beta = x_\beta^{-1}(\overline{K(0,2)})$, then $W_\beta \subset T_\beta \subset V_\beta$. Define $\psi_\beta : M \to \mathbb{R}$ by $\psi_\beta | V_\beta = \psi \circ x_\beta$ and $\psi_\beta | M - T_\beta = 0$ (here ψ is from Lemma 5.3), this glues together, as $\psi \circ x_\beta | V_\beta - T_\beta = 0$.

Now define $\varphi_\alpha : M \to \mathbb{R}$ by

$$\varphi_\alpha = \frac{\sum_{\beta\in J, \lambda(\beta)=\alpha} \psi_\beta}{\sum_{\beta\in J} \psi_\beta}$$

For $p \in M$ let V be an open neighbourhood that meets only $V_{\beta_1}, \ldots, V_{\beta_r}$. The denominator is > 0, as $\psi_\beta | W_\beta = 1$ and $V \subset \bigcup_{i=1}^r W_{\beta_i}$. We also have $\varphi_\alpha | V = 0$ if $\alpha \neq \lambda(\beta_i)$. Hence $(\varphi_\alpha)_{\alpha\in I}$ has the wanted properties. □

5.2. **Orientation.**

Definition 5.4. Let M be a manifold with model space $\mathbb{R}^n$. $\Omega \in \Omega^n(M)$ is called a *volume form* $\iff \quad \forall\, p \in M \quad \Omega(p) \neq 0$.

M is *orientable* $\iff \quad \exists$ volume form on M.

Theorem 5.4. *The following properties are equivalent*

(1) M *is orientable*
(2) $\Omega^n(M)$ *is a free* $\mathcal{C}^\infty(M)$*–module of rank* 1.
(3) $\exists$ *atlas* $x_\alpha : U_\alpha \to \mathbb{R}^n$ *with* $\det \frac{\partial x_\alpha}{\partial x_\beta} > 0$

Proof.

(1) $\Rightarrow$ (2): Let Ω be a volume form, $\omega \in \Omega^n(M)$ arbitrary.
$\exists\ f(p) \in \mathbb{R}$ with $\omega(p) = f(p) \cdot \Omega(p)$, as $0 \neq \Omega(p) \in \bigwedge^n T_p^* M$ is a basis element ($\dim \bigwedge^n T_p^* M = 1$!).
Therefore $f : M \to \mathbb{R}$ is such that $\omega = f \cdot \Omega$ and it remains to be seen that f is differentiable.

Let $x : U \to \mathbb{R}^n$ be a chart, $\omega|U = g \cdot dx^1 \wedge \cdots \wedge dx^n$, $\Omega|U = h \cdot dx^1 \wedge \cdots \wedge dx^n$ with $g, h \in \mathcal{C}^\infty(U)$. As $g = f|U \cdot h$ and $h(p) \neq 0 \quad \forall p \in U$, we have $f|U = g/h$, which is differentiable.

(2) $\Rightarrow$ (1): this is trivial, as a basis element of $\Omega^n(M)$ has to be a volume form.

(1) $\Rightarrow$ (3): Let Ω be a volume form.
Construct an atlas as follows: let $x_\alpha : U_\alpha \to \mathbb{R}^n$ be any atlas with U_α connected. $\Omega = f_\alpha \cdot dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$. We either have $f_\alpha > 0$ or $f_\alpha < 0$ in U_α. In the second case we change the chart x_α to $(-x_\alpha^1, \ldots, x_\alpha^n)$, thus $f_\alpha > 0$ always.

$$\begin{aligned}\Omega|U_\alpha \cap U_\beta &= f_\alpha|U_\alpha \cap U_\beta \cdot dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n \\ &= f_\alpha|U_\alpha \cap U_\beta \cdot \det \frac{\partial x_\alpha}{\partial x_\beta} \cdot dx_\beta^1 \wedge \cdots \wedge dx_\beta^n\end{aligned}$$

Hence

$$\det \frac{\partial x_\alpha}{\partial x_\beta} = \frac{f_\beta|U_\alpha \cap U_\beta}{f_\alpha|U_\alpha \cap U_\beta} > 0$$

(3) $\Rightarrow$ (1): Let $x_\alpha : U_\alpha \to \mathbb{R}^n$ $(\alpha \in I)$ be an atlas with $\det \frac{\partial x_\alpha}{\partial x_\beta} > 0$.
Let $(\varphi_\alpha)_{\alpha \in I}$ be an associated partition of 1. Let $\omega_\alpha : M \to \bigwedge^n T^* M$ be defined by $\omega_\alpha|U_\alpha = \varphi_\alpha|U_\alpha \cdot dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$ and $\omega_\alpha|M - \operatorname{supp} \varphi_\alpha = 0$, this glues together. Now define

$$\begin{aligned}\Omega &: M \longrightarrow \bigwedge^n T^* M \\ \Omega &= \sum_{\alpha \in I} \omega_\alpha\end{aligned}$$

Let $p \in M$, then there $\exists$ open neighbourhood V of p and $J \subset I$ finite such that $\alpha \in J \Leftrightarrow \operatorname{supp} \varphi_\alpha \cap V \neq \varnothing$.

$$\omega_\alpha|U_\alpha \cap U_\beta = \varphi_\alpha|U_\alpha \cap U_\beta \cdot \det \frac{\partial x_\alpha}{\partial x_\beta} \cdot dx_\beta^1 \wedge \cdots \wedge dx_\beta^n$$

As $M = \bigcup_{\alpha \in I} \operatorname{supp} \varphi_\alpha$ we have $V = \bigcup_{\alpha \in I} \operatorname{supp} \varphi_\alpha \cap V \subset \bigcup_{\alpha \in J} \operatorname{supp} \varphi_\alpha \subset \bigcup_{\alpha \in J} U_\alpha$, hence $\exists\ \beta \in J$ with $p \in U_\beta$ and we have

$$\Omega(p) = \big(\sum_{\alpha \in J} \varphi_\alpha(p) \cdot \det \frac{\partial x_\alpha}{\partial x_\beta}(p)\big) \cdot dx_\beta^1 \wedge \cdots \wedge dx_\beta^n \neq 0$$

as the coefficient is > 0. Hence Ω is a volume form. $\square$

Definition 5.5 (orientation)**.** Let M be an orientable manifold. Two volume forms Ω, Ω_1 are *equivalent* $\iff \quad \Omega = f \cdot \Omega_1$ with $f > 0$.

An equivalence class $[\Omega]$ is called an *orientation* of M.

A pair $(M, [\Omega])$ is an *oriented* manifold.

A chart $x : U \to \mathbb{R}^n$ is called *positively oriented* $\iff \quad \Omega|U = f \cdot dx^1 \wedge \cdots \wedge dx^n$ with $f > 0$.

By the theorem on orientable manifolds an orientation is given by an atlas

$$x_\alpha : U_\alpha \longrightarrow \mathbb{R}^n \qquad \text{with } \det \frac{\partial x_\alpha}{\partial x_\beta} > 0$$

The *associated orientation* of such an atlas is that one for which the charts are positively oriented.

5.3. **Integration.**

Theorem 5.5 (Transformation theorem)**.** *Let $U, V \subset \mathbb{R}^n$ be open.*

$$\begin{aligned} f &: U \xrightarrow{\sim} V \qquad \textit{diffeomorphism} \\ g &: V \longrightarrow \mathbb{R} \qquad g \in \mathcal{L}^1(V) \end{aligned}$$

Then $(g \circ f) \cdot |\det Df| \in \mathcal{L}^1(U)$ and

$$\int_V g = \int_U (g \circ f) \cdot |\det Df|$$

Proof. See Lang [5, p. 403].
We apply the result only when g is continuous with compact support, and in that case the proof was given in the class [4, 9.3]. □

Definition 5.6 (Integration)**.** Let $(M, [\Omega])$ be an oriented manifold.

$$\int_M : \Omega^n_{cp}(M) \longrightarrow \mathbb{R}$$

will be defined in two steps.

(1) Let $\omega \in \Omega^n_{cp}(M)$ be such that the support $\operatorname{supp} \omega \subset U$ for a chart $x : U \to \mathbb{R}^n$ positively oriented. Define

$$\int_M \omega = \int_{x(U)} f \circ x^{-1}$$

where $\omega|U = f dx^1 \wedge \cdots \wedge dx^n$.

Independence of the chart:
let $y : U \to \mathbb{R}^n$ be positively oriented $\omega|U = f \cdot \det \frac{\partial x}{\partial y} dy^1 \wedge \cdots \wedge dy^n$. Apply the transformation theorem to the situation

$$y(U) \xrightarrow{x \circ y^{-1}} x(U) \xrightarrow{f \circ x^{-1}} \mathbb{R}$$

$$|\det D(x \circ y^{-1})| = \det \frac{\partial x}{\partial y} \circ y^{-1}$$

$$\int_{x(U)} f \circ x^{-1} = \int_{y(U)} f \circ y^{-1} \cdot \det \frac{\partial x}{\partial y} \circ y^{-1}$$

qed.

(2) Let now $\omega \in \Omega^n_{cp}(M)$ be arbitrary. Choose an atlas of positively oriented charts $x_\alpha : U_\alpha \to \mathbb{R}^n$ and $(\varphi_\alpha)_\alpha$ an associated partition of unity, we define

$$\int_M \omega = \sum_\alpha \int_M \varphi_\alpha \cdot \omega$$

which is a finite sum as the support is compact.

5.4. **Manifolds with boundary.** The whole theory will be repeated with a more general notion of chart Let M be a set, $U \subset M$

$$x : U \longrightarrow H^n \qquad \text{bijection}$$

onto an open subset $x(U) \subset H^n$ will be called chart.
(Here $H^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_n \geq 0\}$ is the *upper half space*).

Definition 5.7 (Manifolds with boundary)**.** A pair $(M, \partial M)$ is called a *manifold with boundary* $\Longleftrightarrow$
M is glued together with an atlas of charts in the new sense by open sets of H^n.

The boundary

$$\partial M = \{p \in M \mid \exists\, x \in C_p \text{ with } x^n(p) = 0\}$$

is a differential geometric invariant, that is the condition $x^n(p) = 0$ is independent of the chosen chart x (see for this Lang [5] or [6]).

$M - \partial M$ is an ordinary manifold without boundary (i.e. in the sense of section 1.2), as well as ∂M, which is of dimension $\dim \partial M = \dim M - 1$.

In particular the upper half space is a manifold with boundary $\partial H^n = \mathbb{R}^{n-1} \times \{0\}$.

Lemma 5.6. *Let $x, y : U \to H^n$ be charts around $p \in \partial M$, then $\frac{\partial x^n}{\partial y^i}(p) = 0$ for $i \neq n$ and $\frac{\partial x^n}{\partial y^n}(p) > 0$.*

Proof. We have for all i that

$$\frac{\partial x^n}{\partial y^i}(p) = D(x^n \circ y^{-1})|_{y(p)}(e_i)$$

Let $y(U) = V \subset H^n$ and $f = x^n \circ y^{-1} : V \to \mathbb{R}$, $a = (a_1, \ldots, a_{n-1}, 0) = y(p)$. We have $f|V \cap H^n = 0$.

For $i \neq n$ we have

$$Df(a)e_i = \lim_{t \to 0} \frac{1}{t}\big(f(\ldots, a_i + t, \ldots, 0) - f(a)\big) = 0$$

as both arguments are in H^n.

For $i = n$ we have

$$Df(a)e_n = \lim_{t \to 0, t > 0} \frac{1}{t} f(a_1, \ldots, a_{n-1}, t) \geq 0$$

as $t > 0$ and $f(a_1, \ldots, a_{n-1}, t) > 0$ and we must have $\frac{\partial x^n}{\partial y^n}(p) \neq 0$, as we have $\det \frac{\partial x}{\partial y}(p) \neq 0$. □

5.4.1. *Induced orientation.*

Theorem 5.7. *Let $(M, \partial M)$ be a manifold with boundary.*

If M is orientable, then ∂M is orientable.

More precisely:
let $x_\alpha : U_\alpha \to H^n$ be an atlas with $\det \frac{\partial x_\alpha}{\partial x_\beta} > 0$, then $y_\alpha : U_\alpha \cap \partial M \to \mathbb{R}^{n-1}$ is an atlas for ∂M with $\det \frac{\partial y_\alpha}{\partial y_\beta} > 0$ where $y_\alpha = ((-1)^n x_\alpha^1, x_\alpha^2, \dots, x_\alpha^{n-1})$.

Remark. This sign is arbitrary, but avoids a sign in Stokes' theorem.

Proof. For $p \in \partial M \cap U_\alpha \cap U_\beta$ we have $\frac{\partial x_\alpha}{\partial x_\beta}(p) \in GL(\mathbb{R}^n)$ with the matrix $\big(\frac{\partial x_\alpha^i}{\partial x_\beta^j}(p)\big)_{ij}$. In the last row $(i = n)$ there are all zeros except for the last column $\frac{\partial x_\alpha^n}{\partial x_\beta^n}(p) > 0$. Taking determinants we have

$$\det \frac{\partial x_\alpha}{\partial x_\beta}(p) = (-1)^n \cdot (-1)^n \cdot \det \frac{\partial y_\alpha}{\partial y_\beta}(p) \cdot \frac{\partial x_\alpha^n}{\partial x_\beta^n}(p)$$

and this implies our assertion. □

Definition 5.8. Let $(M, [\Omega], \partial M)$ be an oriented manifold with boundary. The orientation defined by the theorem above is called *induced orientation* and will be denoted by $[\partial M]$.

When the local representation in a chart $x : U \to H^n$ of Ω is

$$\Omega | U = dx^1 \wedge \cdots \wedge dx^n$$

then the induced volume form is

$$\partial\Omega | U \cap \partial M = (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}$$

Remark. The inclusion

$$\iota : \partial M \hookrightarrow M$$

is differentiable.

5.5. **The Theorem of Stokes.**

Theorem 5.8 (Theorem of Stokes)**.** *Let $(M, [\Omega], \partial M)$ be an oriented manifold with boundary, $\iota : \partial M \hookrightarrow M, \omega \in \Omega^{n-1}(M)$ $(n = \dim M)$. Then*

$$\int_M d\omega = \int_{\partial M} \iota^*(\omega)$$

Here $\iota^*(\omega)$ denotes the pulled back differential form on the boundary. More generally

5.5.1. *Pull back of differential forms.*

Let $f : N \to M$ be differentiable, we define $\mathbb{R}$–linear maps $f^* : \Omega^r(M) \to \Omega^r(N)$ by the following construction:

$$T_p f : T_p N \longrightarrow T_{f(p)} M$$

This induces by multilinear algebra a map $\bigwedge^r T_p^* f : \bigwedge^r T_{f(p)}^* M \longrightarrow \bigwedge^r T_p^* N$
$f^*(\omega)_p = \bigwedge^r T_p^* f(\omega_{f(p)})$ for $p \in N$ and $\omega \in \Omega^r(M)$.

$$\begin{tikzcd}
\bigwedge^r T^* N & \bigwedge^r T^* M \arrow[l] \\
N \arrow[u, "f^*(\omega)"] \arrow[r, "f"'] & M \arrow[u, "\omega"']
\end{tikzcd}$$

For $g : M \to \mathbb{R}$ we get by the chain rule

$$T_p(g \circ f) = T_{f(p)} g \circ T_p f$$
$$d(g \circ f)_p = dg_{f(p)} \circ T_p f = T_p^* f(dg_{f(p)})$$

With this we can easily get the local representation of a pulled back differential form:

Let $y : V \to \mathbb{R}^k$ a local chart of N, $x : U \to \mathbb{R}^n$ a local chart of M such that $f(V) \subset U$, let $\omega|U = \sum g_{i_1,\dots,i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$ then we get

$$f^*(\omega)|V = \sum g_{i_1,\dots,i_r} \circ f|V \cdot d(x^{i_1} \circ f) \wedge \dots \wedge d(x^{i_r} \circ f)|V$$

where you can further substitute

$$d(x^i \circ f) = \sum_{j=1}^{k} \frac{\partial(x^i \circ f)}{\partial y^j} dy^j$$

Pull back is compatible with exterior differentiation, as is obvious from the local expression:

$$d \circ f^* = f^* \circ d$$

Proof of Stokes' theorem. We will proceed in several steps. Let us first remark that the integrals are defined, i.e. the differential forms have compact support:

$$\operatorname{supp} d\omega \subset \operatorname{supp} \omega \quad \text{and} \quad \operatorname{supp} \iota^*(\omega) \subset \operatorname{supp} \omega \cap \partial M$$

The case $\operatorname{supp} \omega \subset U$ for a chart $x : U \to H^n$.

If $U \cap \partial M = \varnothing$ then $\iota^*(\omega) = 0$ and $\int_{\partial M} \iota^*(\omega) = 0$.

$\exists$ positiv oriented chart $x : U \xrightarrow{\sim} (0,1)^n$, let

$$\omega|U = \sum_{i=1}^{n} f_i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

then

$$d\omega|U = (\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x^i}) dx^1 \wedge \dots \wedge dx^n$$

and the integral is

$$\int_M d\omega = \sum (-1)^{i-1} \int_{(0,1)^n} D_i(f_i \circ x^{-1}) = 0$$

as the support $\operatorname{supp}(f_i \circ x^{-1}) \subset [\varepsilon, 1-\varepsilon]^n$ for a suitable $\varepsilon > 0$ and by Fubini

$$\int_{(0,1)^n} D_i(f_i \circ x^{-1}) = \int_{(0,1)^{n-1}} \big(f_i \circ x^{-1}(\dots, 1-\varepsilon, \dots) - f_i \circ x^{-1}(\dots, \varepsilon, \dots)\big) = 0$$

Now assume that $U \cap \partial M \neq \varnothing$:

$\exists$ positiv oriented chart $x : U \xrightarrow{\sim} (0,1)^{n-1} \times [0,1)$, let $\omega|U$ be as above, and look thru the calculation above. We now have that the support $\operatorname{supp}(f_i \circ x^{-1}) \subset [\varepsilon, 1-\varepsilon]^{n-1} \times [0, 1-\varepsilon]$ and in the calculation of the integral there will now be left the last term

$$\int_M d\omega = (-1)^{n-1} \int_{(0,1)^{n-1}} -f_n \circ x^{-1}(\dots, 0) = (-1)^n \int_{(0,1)^{n-1}} f_n \circ x^{-1}(\dots, 0)$$

On the other hand we now have for the restricted form on the boundary

$$\iota^*(\omega)|U \cap \partial M = f_n \circ \iota \cdot dx^1 \wedge \dots \wedge dx^{n-1} = (-1)^n f_n \circ \iota \cdot dy^1 \wedge \dots \wedge dy^{n-1}$$

$$\int_{\partial M} \iota^*(\omega) = (-1)^n \int_{y(U)} f_n \circ y^{-1} = (-1)^n \int_{(0,1)^{n-1}} f_n \circ x^{-1}(\ldots, 0)$$

At last we consider the general case when the support of the form is larger than any domain of a chart. We choose a partition of unity and make use of $\sum \varphi_\alpha = 1$ implies that $\sum d\varphi_\alpha = 0$

$$\int_M d\omega = \sum_\alpha \int_M \varphi_\alpha \cdot d\omega = \sum_\alpha \int_M \big(\varphi_\alpha \cdot d\omega + d\varphi_\alpha \wedge \omega\big) = \sum_\alpha \int_M d(\varphi_\alpha \cdot \omega)$$

and for the integral on the boundary we have

$$\int_{\partial M} \iota^*(\omega) = \sum_\alpha \int_{\partial M} \varphi_\alpha \circ \iota \cdot \iota^*(\omega) = \sum_\alpha \int_{\partial M} \iota^*(\varphi_\alpha \cdot \omega)$$

and by the previous step we see that the integrals on the right are equal. □

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