

INNER PLANETS OF SOLAR SYSTEM

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For Carmen

ABSTRACT. This paper provides a visualisation of the four inner planets of the solar system.

PREFACE

In this paper I visualise the *orbits* of the *inner* planets: *Mercury*, *Venus*, *Earth* and *Mars*. These orbits are *planar curves* in *space*. Since KEPLER (1571 – 1630) we know the shape of these curves: they are *ellipses*, which we need to put into their correct relative positions and orientation in different orbital planes in space.

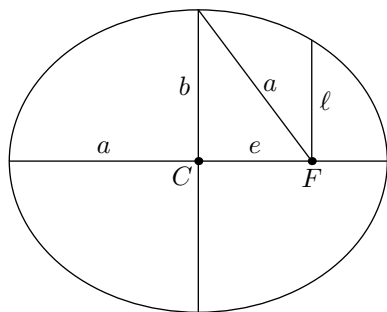
The visualisation is provided by figures drawn with METAPOST, where I disregard the inclination of the orbits towards the ecliptic. The presentation is kept at high school level with an emphasis of introducing ideas in a conceptual manner, though not losing sight of computation.

Berlin, April 24, 2010

1. ORBITS OF PLANETS

1.1. **Inside the orbital plane.** As a mathematical warm-up we will start with some planar geometry. We recall the *fundamental invariants* determining the shape of an ellipse, as in section *Conics* of COXETER [1, §8.4], or in more detail in [3, §1.2].

The fundamental invariants of ellipses are the *numerical eccentricity* ε and the *semilatus rectum* ℓ . The shape (*ovalness*) is determined by ε , the size by ℓ . In astronomical context rather than ℓ the *semimajor axis* a is given.



The entities in the figure are related as follows

$$\begin{aligned} a &= \frac{\ell}{1 - \varepsilon^2} & b &= \frac{\ell}{\sqrt{1 - \varepsilon^2}} \\ e &= \varepsilon \cdot a & a^2 &= b^2 + e^2 \end{aligned}$$

b is the *semiminor axis*, e is the *linear eccentricity* see [3, (1.9), (1.12), (1.14), (1.15)]

The equation of the ellipse in the coordinate system of the axes is given by [3, (1.13)]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

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When this is written $x^2 + (ay/b)^2 = a^2$, the ellipse appears as a circle of radius a

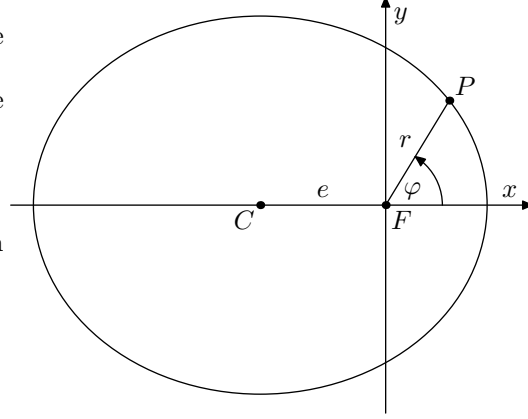
scaled down in the y -direction by the scaling factor $b/a = \sqrt{1 - \varepsilon^2}$.

When the focus F is the origin the ellipse equation reads

$$(1) \quad (x + e)^2 + \frac{y^2}{1 - \varepsilon^2} = a^2$$

From this one we derive the equation in polar coordinates

$$(2) \quad r = \frac{\ell}{1 + \varepsilon \cos \varphi}$$



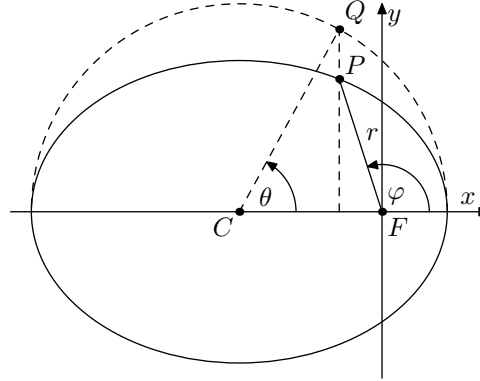
Proof. By (1)

$$0 = (1 - \varepsilon^2)(x^2 + 2ex + e^2 - a^2) + y^2 = x^2 + y^2 - \varepsilon^2 x^2 + 2(1 - \varepsilon^2)ex - (1 - \varepsilon^2)a^2 = r^2 - \varepsilon^2 x^2 + 2\ell \varepsilon x - \ell^2 = r^2 - (\varepsilon x - \ell)^2 = (r + \varepsilon x - \ell)(r - \varepsilon x + \ell)$$

giving us two possible polar equations for the right focal point, $r(1 + \varepsilon \cos \varphi) = \ell$ and $r(\varepsilon \cos \varphi - 1) = \ell$; the latter only gives points for $\varepsilon > 1$ (*hyperbola*) as $r > 0$. \square

In the next figure we let the point $P = (x, y) = (r \cos \varphi, r \sin \varphi)$ move on the ellipse and consider its twin $Q = (x, ay/b)$ on the circle of radius a around $C = (-e, 0)$

$$(3) \quad \begin{aligned} \sin \theta &= \frac{\sqrt{1 - \varepsilon^2} \sin \varphi}{1 + \varepsilon \cos \varphi} \\ \cos \theta &= \frac{\cos \varphi + \varepsilon}{1 + \varepsilon \cos \varphi} \\ \sin \varphi &= \frac{\sqrt{1 - \varepsilon^2} \sin \theta}{1 - \varepsilon \cos \theta} \\ \cos \varphi &= \frac{\cos \theta - \varepsilon}{1 - \varepsilon \cos \theta} \\ \tan \frac{\varphi}{2} &= \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \tan \frac{\theta}{2} \end{aligned}$$



Polar coordinates of Q imply $a \sin \theta = ay/b$ and $a \cos \theta = x + e$, so by (2):

$$\begin{aligned} \sin \theta &= \frac{y}{b} = \frac{r \sin \varphi}{b} = \frac{\sqrt{1 - \varepsilon^2} \sin \varphi}{1 + \varepsilon \cos \varphi} \\ \cos \theta &= \frac{x + e}{a} = \frac{r \cos \varphi + \varepsilon a}{a} = \frac{(1 - \varepsilon^2) \cos \varphi}{1 + \varepsilon \cos \varphi} + \varepsilon = \\ &= \frac{\cos \varphi - \varepsilon^2 \cos \varphi + \varepsilon + \varepsilon^2 \cos \varphi}{1 + \varepsilon \cos \varphi} = \frac{\cos \varphi + \varepsilon}{1 + \varepsilon \cos \varphi} \end{aligned}$$

giving the first two relations in (3). The others are immediate as well, for example

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{\sqrt{1 - \varepsilon^2} \sin \varphi}{1 + \varepsilon \cos \varphi + \cos \varphi + \varepsilon} = \frac{\sqrt{1 - \varepsilon^2}}{1 + \varepsilon} \frac{\sin \varphi}{1 + \cos \varphi}$$

We also note the formula (directly implied by (3)):

$$(4) \quad r = a(1 - \varepsilon \cos \theta) = \frac{\ell}{1 + \varepsilon \cos \varphi}$$

The differential of (4) gives $dr = a\varepsilon \sin \theta d\theta$; similarly from (2) we obtain

$$\begin{aligned} r\varepsilon \sin \varphi d\varphi &= (1 + \varepsilon \cos \varphi) dr = a\varepsilon(1 + \varepsilon \cos \varphi) \sin \theta d\theta = && \text{by (3)} \\ &= a\sqrt{1 - \varepsilon^2} \varepsilon \sin \varphi d\theta \end{aligned}$$

hence

$$(5) \quad (1 - \varepsilon \cos \theta) d\varphi = \sqrt{1 - \varepsilon^2} d\theta$$

1.2. Celestial mechanics. Let V designate our 3-dimensional vector space with the sun in the origin. Let the vector $v \in V$ point to the planet under consideration. By NEWTON's *gravitational law* we have

$$(6) \quad \ddot{v} = -\frac{\alpha}{r^3}v, \quad r = |v|,$$

where $\alpha = G \cdot (M + m)$, G is the *gravitational constant*, M is the mass of the sun and m that of the planet.

Remark. In fact, we have put the barycentre at rest, the planet is at position $\frac{M}{M+m}v$ and the sun is at position $-\frac{m}{M+m}v$. For our illustrations this will make no real difference, as the barycentres of all inner planets lie deep inside the sun. Anyway, a point representing the sun in a figure is about five times larger in ratio to the real sun.

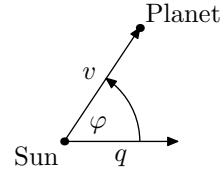
The vector $c = v \times \dot{v}$ is constant: $\dot{c} = v \times \ddot{v} = 0$, as \ddot{v} is parallel to v by (6). Let $L = \mathbb{R}c$ be the line spanned by c and $P = L^\perp$ be the plane perpendicular to L , so $V = L \oplus P$. We have just seen that both v and \dot{v} lie in the plane P for all time: $v, \dot{v} \in P$, the planetary movement is *planar*.

Now we investigate the plane vector $q = \dot{v} \times c - \frac{\alpha}{r}v$. Differentiation gives

$$\dot{q} = \ddot{v} \times c - \frac{\alpha}{r} \dot{v} + \frac{\alpha \dot{r}}{r^2}v = -\frac{\alpha}{r^3}v \times c - \frac{\alpha}{r} \dot{v} + \frac{\alpha \dot{r}}{r^2}v = 0$$

as $v \times c = v \times (v \times \dot{v}) = r\dot{r}v - r^2\dot{v}$ and $v \cdot \dot{v} = r\dot{r}$ by deriving $v \cdot v = r^2$. So, q is an invariable vector in the orbital plane and we let φ be the angle between v and q , having thus introduced *polar coordinates* $r = |v|$, $\varphi = \angle(q, v)$ in P .

Now $q \cdot v = v \cdot (\dot{v} \times c) - \alpha r = (v \times \dot{v}) \cdot c - \alpha r = c \cdot c - \alpha r$ or $r(\alpha + |q| \cos \varphi) = |c|^2$, which takes the form (2) with $\varepsilon = |q|/\alpha$ and $\ell = |c|^2/\alpha$. This is KEPLER's first law.



In case $\varepsilon < 1$ (ellipse) the minimal distance of the planet from sun $r_{min} = \frac{\ell}{1+\varepsilon}$ is at $\varphi = 0$ (*perihelion*) and the maximal distance $r_{max} = \frac{\ell}{1-\varepsilon}$ is at $\varphi = \pi$ (*aphelion*). The angle φ is also called the *true anomaly* and the angle θ of the previous section 1.1 is called the *eccentric anomaly*.

Let us introduce an orthonormal frame (e_0, e_1, e_2) in V such that $L = \mathbb{R}e_0$ and $P = \mathbb{R}e_1 + \mathbb{R}e_2$, with e_1 pointing to the perihelion (so $q = |q|e_1$) and $e_2 = e_0 \times e_1$. Then $v = xe_1 + ye_2$ and $\dot{v} = \dot{x}e_1 + \dot{y}e_2$, hence $c = v \times \dot{v} = (x\dot{y} - \dot{x}y)e_0$. In polar coordinates we have

$$\begin{aligned} x &= r \cos \varphi & \dot{x} &= \dot{r} \cos \varphi - r\dot{\varphi} \sin \varphi \\ y &= r \sin \varphi & \dot{y} &= \dot{r} \sin \varphi + r\dot{\varphi} \cos \varphi \end{aligned}$$

and $|c| = x\dot{y} - \dot{x}y = r^2\dot{\varphi}$. This is the constancy of the *orbital angular momentum* of KEPLER's second law:

$$(7) \quad \sqrt{\ell\alpha} dt = r^2 d\varphi$$

The movement of planets on ellipses is not uniform, as r is not constant for $\varepsilon > 0$. The correct relation is the famous KEPLER equation (8) relating time and position. The proof also gives us KEPLER's third law (9).

$$(8) \quad \theta - \varepsilon \sin \theta = \frac{2\pi}{T}(t - t_0)$$

$$(9) \quad \frac{a^3}{T^2} = \frac{\alpha}{4\pi^2} = \frac{G(M + m)}{4\pi^2}$$

where T is the revolution time of the planet around the sun.

Proof. Substituting (4) in (7) $\sqrt{\ell\alpha} dt = a^2(1 - \varepsilon \cos \theta)^2 d\varphi = a^2\sqrt{1 - \varepsilon^2}(1 - \varepsilon \cos \theta) d\theta$ by (5). We get by integration (where t_0 corresponds to the time at perihelion):

$$\sqrt{\frac{\alpha}{a^3}} \int_{t_0}^t dt = \sqrt{\frac{\alpha}{a^3}}(t - t_0) = \int_0^\theta (1 - \varepsilon \cos \theta) d\theta = \theta - \varepsilon \sin \theta$$

After one revolution $\theta = 2\pi$ and $t - t_0 = T$, completing the proof. \square

2. ORBITAL ELEMENTS

The orbital plane of the Earth is called ecliptic, the position of the sun will be the origin and the x -axis points to the vernal point Υ . I disregard the inclination of the other planets against the ecliptic.

For our purpose the approximate positions in STANDISH [4] are sufficient. The calculations of the elements, in particular for the *eccentric anomaly* θ is done in the program `solar.c`, see appendix A.

The orbits are drawn as rotated ellipses with METAPOST with the sun in the origin. The values for the *true anomaly* are taken from the output of the `solar` program. The following visualisation of the orbits of the inner planets at their location on October 13, 2013 is generated by the METAPOST program:

```
% -----
%      Abbildung 5 -- inner planets
% -----

beginfig(5);
def orbit(expr ra,rb,an)=
  (fullcircle xscaled 2ra yscaled 2rb shifted -(ra+-+rb,0) rotated an)
enddef;
path vernal,part;
part = ((-0.25,0.25){2up+left}..(-0.2,0.5){right}.. (0,0){down});
vernal = part & reverse part reflectedabout ((0,0),up);
v=.4mm;
picture sun,mercury,venus,earth,mars;

draw fullcircle scaled 10v;
fill fullcircle scaled 2v;
sun = currentpicture;
clearit;

draw fullcircle scaled 5v shifted (0,.5v);
draw halfcircle scaled 5v rotated 180 shifted (0,5.5v);
draw (-2v,-4v)--(2v,-4v);
draw (0,-2v)--(0,-6v);
mercury = currentpicture;
clearit;
```

```

draw fullcircle scaled 5v shifted (0,.5v);
draw (-2v,-4v)--(2v,-4v);
draw (0,-2v)--(0,-6v);
venus = currentpicture;
clearit;

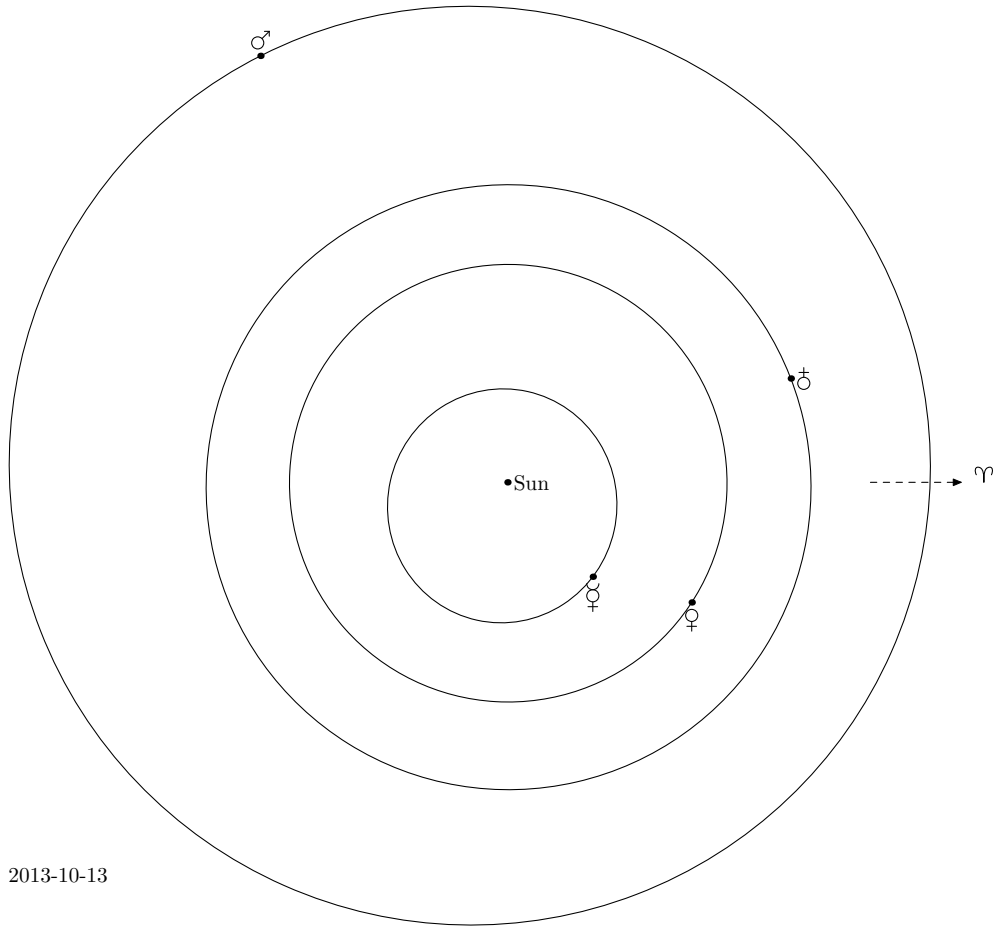
draw fullcircle scaled 5v shifted (0,-2.5v);
draw (-2v,2v)--(2v,2v);
draw (0,0)--(0,4v);
earth = currentpicture;
clearit;

path q[];
draw fullcircle scaled 5v shifted (0,-2.5v);
z10=(2.5/sqrt(2)*v,2.5/sqrt(2)*v);
z20=z10+(3v,3v);
q0=z10--z20;
q1=(x20-2v,y20)--z20;
q2=(x20,y20-2v)--z20;
draw q0 shifted (0,-2.5v);
draw q1 shifted (0,-2.5v);
draw q2 shifted (0,-2.5v);
mars = currentpicture;
clearit;

path p[];
z0=(0,0);
a1=3.871u; b1=0.9786*a1;
a2=7.233u; b2=0.99998*a2;
a3=10u; b3=0.99986*a3;
a4=15.237u; b4=0.996*a4;
% Sun
% Mercury
% Venus
% Earth
% Mars
% *** Output of solar 2013-10-13: ***
%julian : 2456579
%Mercury: -47.89, Perihel: 77.48
%Venus : -33.10, Perihel: 131.78
%Earth : 20.12, Perihel: 102.97
%Mars : 120.08, Perihel: -23.86
label.lft(btex 2013-10-13 etex,(-13u,-13u));
v1=-47.89; v2=-33.10; v3=20.12; v4=120.08;
w1=77.48; w2=131.78; w3= 102.97; w4=-23.86;
for i=1 upto 4:
  p[i]=orbit(a[i],b[i],w[i]);
  draw p[i];
endfor
for i=1 upto 4:
  z[i]=2a[i]*dir v[i];
  p[4+i]=z0--z[i];
  z[4+i]=p[i] intersectionpoint p[4+i];
endfor
dotlabel.bot(mercury,z5);
dotlabel.bot(venus,z6);
dotlabel.rt(earth,z7);
dotlabel.top(mars,z8);
draw sun;
z9=(15u,0);
drawarrow .8z9--z9 dashed evenly; % direction to vernal point

```

```
draw vernal scaled u shifted 1.05z9;
endfig;
% -----
```



APPENDIX A. THE PROGRAM `SOLAR.C`

The program `solar.c` calculates the true anomaly of the inner planets from a date. The calculation of the *julian* day number has been taken from the article *On calendar formulas* [2]. The routine `theta` solves the KEPLER equation (8).

```
//
```

```
/* -----*
```

```
Module:    solar.c
```

```
Description:
```

```
Calculating the true anomaly of inner planets
```

```
For details: http://berndt-schwerdtfeger.de/sol/solar.pdf
```

```
Input:  yyyy-mm-dd (-3000 <= yyyy <= +3000)
```

```
Output: direction of planets Mercury, Venus, Earth, Mars
```

Subroutines: theta, leap, qu

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 limitations under the License.

-----*/

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

// set constants - Keplerian elements from Standish

char *planet[] = {"Mercury", "Venus ", "Earth ", "Mars  "};
int em[13] = {0,0,31,59,90,120,151,181,212,243,273,304,334};
double e[4] = {0.20563661, 0.00676399, 0.01673163, 0.09336511};
double de[4]= {0.00002123,-0.00005107,-0.00003661, 0.00009149};
double L[4] = {252.25166724, 181.97970850, 100.46691572, -4.56813164};
double dL[4]= {149472.67486623, 58517.81560260, 35999.37306329, 19140.29934243};
double w[4] = {77.45771895, 131.76755713, 102.93005885, -23.91744784};
double dw[4]= { 0.15940013, 0.05679648, 0.31795260, 0.45223625};
#define PI 3.14159265358979324
#define DEG (180.0/PI)
#define RAD (PI/180.0)

// prototype statements for functions

double theta(double,double); // theta - eps*sin theta = m
int leap(long); // = 1 if year is a leap year
long qu(int,int); // = [a/b] Gaussklammer

// -----
// main program
// -----

int main(int argc, char *argv[]){

    if (argc == 1 || *argv[1] == '?'){

        printf("\nsolar $Revision: 387 $, $Date: 2014-03-22 17:59:59 +0100 (Sa, 22 Mrz 2014) $ \n");
        printf("-- Copyright (C) 2010 Berndt E. Schwerdtfeger -- \n\n");
        printf("\nTrue anomaly of Mercury, Venus, Earth and Mars \n");
        printf(" Input:      yyyy-mm-dd (yyyy in range -3000 - 3000) \n");
        return EXIT_SUCCESS; // end the program
    }
}
```

```

} // argc > 1 here

int i,d,m,y;
long j,x;
double T, c[4], f[4], l[4], t[4], v[4];

sscanf(argv[1],"%d-%d-%d",&y,&m,&d);

if (argc > 2) {
    printf("Too many parameters !\n");
    return EXIT_FAILURE; // end the program
}
if (y<-3000 || y>3000) {
    printf("Error: outside allowed range -3000 <= year <= +3000 !\n\n");
    return EXIT_FAILURE; // end the program
}
if (y==1582 && m==10 && d>4 && d<15) {
    printf("This date is invalid, it did never exist !\n");
    return EXIT_FAILURE; // end the program
}

if (y < 1582 || y == 1582 && m*100+d < 1005) // Julian era
    j = 1721423 + d + em[m] + (y-1)*365 + qu(y-1,4);
else { // Gregorian era
    x = y - 2001;
    j = 2451910 + d + em[m] + x*365 + qu(x,4) - qu(x,100) + qu(x,400);
}
if (m>2) j+=leap(y); // correction for leap year

x = j - 2451545; // days since 2000-01-01
T = (double)x/(double)36525; // julian centuries

for (i=0;i<4;i++) {
    c[i] = e[i]+de[i]*T; // eccentricity
    l[i] = L[i]+dL[i]*T; // mean longitude
    v[i] = w[i]+dw[i]*T; // longitude of perihelion
    f[i] =(l[i] - v[i])*RAD; // mean anomaly
    t[i] = theta(f[i],c[i]); // eccentric anomaly
    f[i] = 2*atan(sqrt((1+c[i])/(1-c[i]))*tan(t[i]/2))*DEG+v[i]; // true a.
}

printf("julian : %ld\n",j);
for (i=0;i<4;i++) printf("%s: %.2f, Perihel: %.2f\n", planet[i], f[i],v[i]);

return EXIT_SUCCESS; // exit the program
}

// -----
// subroutine: theta, leap, qu
// -----

double theta(double m, double e){ // solve Kepler equation
    double t, d, dt;

    while (m > PI) m-=2*PI ; // normalize m ..
    while (m < -PI) m+=2*PI ; // .. to range -PI <= m <= PI

```



```

t = m + e*sin(m);

do
{ d = m - (t - e*sin(t));
  dt= d/(1-e*cos(t));
  t += dt;
} while (fabs(dt) > 0.0001);

return t;
}

// -----
int leap(long y){

int i = 0;
if (y%4 == 0)
  i = 1;
if (y > 1582 && y%100 == 0 && y%400 != 0)
  i = 0;
return i;
}

// -----
long qu(int a, int b){

long x = a/b;
if (a < 0) // for negative numerator ..
  x -= (a%b != 0); // .. if remainder, subtract 1
return x;
}
//

```

REFERENCES

- [1] Harold Scott MacDonald Coxeter, *Introduction to Geometry*, 2nd ed., John Wiley & Sons, 1969.
- [2] Berndt E. Schwerdtfeger, *On calendar formulas* (2010), available at <http://berndt-schwerdtfeger.de/wp-content/uploads/pdf/calj.pdf>.
- [3] ———, *Invariants of Curves of second order* (2013), available at <http://berndt-schwerdtfeger.de/wp-content/uploads/pdf/c2.pdf>.
- [4] Erland Myles Standish, *Keplerian Elements for Approximate Positions of the Major Planets*, available at http://ssd.jpl.nasa.gov/txt/aprx_pos_planets.pdf.