INNER PLANETS OF SOLAR SYSTEM

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For Carmen

ABSTRACT. This paper provides a visualization of the four inner planets of the solar system.

PREFACE

In this paper I visualize the *orbits* of the *inner* planets: *Mercury*, *Venus*, *Earth* and *Mars*. These orbits are *planar curves* in *space*. Since KEPLER (1571 – 1630) we know the shape of these curves: they are *ellipses*, which we need to put into their correct relative positions and orientation in different orbital planes in space.

The visualization is provided by figures drawn with METAPOST, where I disregard the inclination of the orbits towards the ecliptic. The presentation is kept at high school level with an emphasis of introducing ideas in a conceptual manner, though not losing sight of computation.

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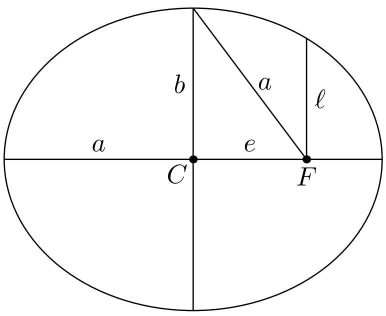
1. ORBITS OF PLANETS

1.1. **Inside the orbital plane.** As a mathematical warm-up we will start with some planar geometry. We recall the *fundamental invariants* determining the shape of an ellipse, as in section *Conics* of COXETER [1, §8.4], or in more detail in [3, §1.2].

The fundamental invariants of ellipses are the *numerical eccentricity* ε and the *semilatus rectum* ℓ. The shape (*ovalness*) is determined by ε, the size by ℓ. In astronomical context rather than ℓ the *semimajor axis* a is given.



The entities in the figure are related as follows

$$a = \frac{\ell}{1-\varepsilon^2} \qquad b = \frac{\ell}{\sqrt{1-\varepsilon^2}}$$

$$e = \varepsilon \cdot a \qquad a^2 = b^2 + e^2$$

b is the *semiminor axis*, e is the *linear eccentricity* see [3, (1.9), (1.12), (1.14), (1.15)]

The equation of the ellipse in the coordinate system of the axes is given by [3, (1.13)]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

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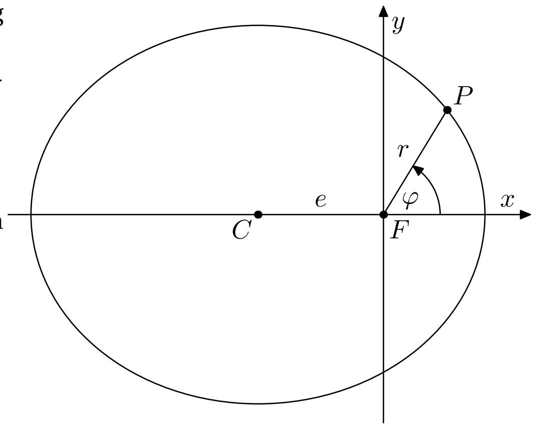
When this is written $x^2+(ay/b)^2=a^2$, the ellipse appears as a circle of radius a *scaled down* in the y-direction by the scaling factor $b/a=\sqrt{1-\varepsilon^2}$.

When the *focus* F is the origin the ellipse equation reads



(1) $$(x+e)^2+\frac{y^2}{1-\varepsilon^2}=a^2$$

From this one we derive the equation in *polar coordinates*

(2) $$r=\frac{\ell}{1+\varepsilon\cos\varphi}$$

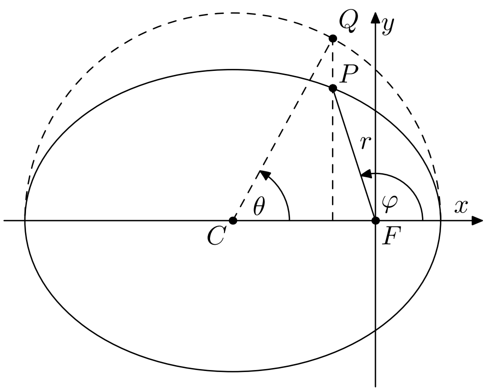
Proof. By (1)

$$0=(1-\varepsilon^2)(x^2+2ex+e^2-a^2)+y^2=x^2+y^2-\varepsilon^2x^2+2(1-\varepsilon^2)ex-(1-\varepsilon^2)b^2=$$
$$=r^2-\varepsilon^2x^2+2\ell\varepsilon x-\ell^2=r^2-(\varepsilon x-\ell)^2=(r+\varepsilon x-\ell)(r-\varepsilon x+\ell)$$

giving us two possible polar equations for the right focal point, $r(1+\varepsilon\cos\varphi)=\ell$ and $r(\varepsilon\cos\varphi-1)=\ell$; the latter only gives points for $\varepsilon>1$ (*hyperbola*) as $r>0$. □

In the next figure we let the point $P=(x,y)=(r\cos\varphi,r\sin\varphi)$ move on the ellipse and consider its twin $Q=(x,ay/b)$ on the circle of radius a around $C=(-e,0)$

(3) $$\begin{aligned}\sin\theta&=\frac{\sqrt{1-\varepsilon^2}\sin\varphi}{1+\varepsilon\cos\varphi}\\ \cos\theta&=\frac{\cos\varphi+\varepsilon}{1+\varepsilon\cos\varphi}\\ \sin\varphi&=\frac{\sqrt{1-\varepsilon^2}\sin\theta}{1-\varepsilon\cos\theta}\\ \cos\varphi&=\frac{\cos\theta-\varepsilon}{1-\varepsilon\cos\theta}\\ \tan\frac{\varphi}{2}&=\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\theta}{2}\end{aligned}$$



Polar coordinates of Q imply $a\sin\theta=ay/b$ and $a\cos\theta=x+e$, so by (2):

$$\sin\theta=\frac{y}{b}=\frac{r\sin\varphi}{b}=\frac{\sqrt{1-\varepsilon^2}\sin\varphi}{1+\varepsilon\cos\varphi}$$
$$\cos\theta=\frac{x+e}{a}=\frac{r\cos\varphi+\varepsilon a}{a}=\frac{(1-\varepsilon^2)\cos\varphi}{1+\varepsilon\cos\varphi}+\varepsilon=$$
$$=\frac{\cos\varphi-\varepsilon^2\cos\varphi+\varepsilon+\varepsilon^2\cos\varphi}{1+\varepsilon\cos\varphi}=\frac{\cos\varphi+\varepsilon}{1+\varepsilon\cos\varphi}$$

giving the first two relations in (3). The others are immediate as well, for example

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}=\frac{\sqrt{1-\varepsilon^2}\sin\varphi}{1+\varepsilon\cos\varphi+\cos\varphi+\varepsilon}=\frac{\sqrt{1-\varepsilon^2}}{1+\varepsilon}\,\frac{\sin\varphi}{1+\cos\varphi}$$

We also note the formula (directly implied by (3)):

(4) $$r=a(1-\varepsilon\cos\theta)=\frac{\ell}{1+\varepsilon\cos\varphi}$$

The differential of (4) gives $dr = a\varepsilon \sin\theta\, d\theta$; similarly from (2) we obtain

$$r\varepsilon \sin\varphi\, d\varphi = (1+\varepsilon\cos\varphi)\, dr = a\varepsilon(1+\varepsilon\cos\varphi)\sin\theta\, d\theta = \qquad \text{by (3)}$$
$$= a\sqrt{1-\varepsilon^2}\,\varepsilon\sin\varphi\, d\theta$$

hence

$$(1-\varepsilon\cos\theta)\, d\varphi = \sqrt{1-\varepsilon^2}\, d\theta \tag{5}$$

1.2. **Celestial mechanics.** Let V designate our 3-dimensional vector space with the sun in the origin. Let the vector $v \in V$ point to the planet under consideration. By NEWTON's *gravitational law* we have

$$\ddot{v} = -\frac{\alpha}{r^3}v, \qquad r = |v|, \tag{6}$$

where $\alpha = G\cdot(M+m)$, G is the *gravitational constant*, M is the mass of the sun and m that of the planet.

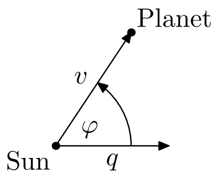
Remark. In fact, we have put the barycentre at rest, the planet is at position $\frac{M}{M+m}v$ and the sun is at position $-\frac{m}{M+m}v$. For our illustrations this will make no real difference, as the barycentres of all inner planets lie deep inside the sun. Anyway, a point representing the sun in a figure is about five times larger in ratio to the real sun.

The vector $c = v\times\dot{v}$ is constant: $\dot{c} = v\times\ddot{v} = 0$, as $\ddot{v}$ is parallel to v by (6). Let $L = \mathbf{R}c$ be the line spanned by c and $P = L^{\perp}$ be the plane perpendicular to L, so $V = L\oplus P$. We have just seen that both v and $\dot{v}$ lie in the plane P for all time: $v, \dot{v} \in P$, the planetary movement is *planar*.

Now we investigate the plane vector $q = \dot{v}\times c - \frac{\alpha}{r}v$. Differentiation gives

$$\dot{q} = \ddot{v}\times c - \frac{\alpha}{r}\dot{v} + \frac{\alpha\dot{r}}{r^2}v = -\frac{\alpha}{r^3}v\times c - \frac{\alpha}{r}\dot{v} + \frac{\alpha\dot{r}}{r^2}v = 0$$

as $v\times c = v\times(v\times\dot{v}) = r\dot{r}v - r^2\dot{v}$ and $v\cdot\dot{v} = r\dot{r}$ by deriving $v\cdot v = r^2$. So, q is an invariable vector in the orbital plane and we let φ be the angle between v and q, having thus introduced *polar coordinates* $r = |v|$, $\varphi = \angle(q, v)$ in P.



Now $q\cdot v = v\cdot(\dot{v}\times c) - \alpha r = (v\times\dot{v})\cdot c - \alpha r = c\cdot c - \alpha r$ or $r(\alpha + |q|\cos\varphi) = |c|^2$, which takes the form (2) with $\varepsilon = |q|/\alpha$ and $\ell = |c|^2/\alpha$. This is KEPLER's first law.

In case $\varepsilon < 1$ (ellipse) the minimal distance of the planet from sun $r_{min} = \frac{\ell}{1+\varepsilon}$ is at $\varphi = 0$ (*perihelion*) and the maximal distance $r_{max} = \frac{\ell}{1-\varepsilon}$ is at $\varphi = \pi$ (*aphelion*). The angle φ is also called the *true anomaly* and the angle θ of the previous section 1.1 is called the *eccentric anomaly*.

Let us introduce an orthonormal frame (e_0, e_1, e_2) in V such that $L = \mathbf{R}e_0$ and $P = \mathbf{R}e_1 + \mathbf{R}e_2$, with e_1 pointing to the perihelion (so $q = |q|e_1$) and $e_2 = e_0\times e_1$. Then $v = xe_1 + ye_2$ and $\dot{v} = \dot{x}e_1 + \dot{y}e_2$, hence $c = v\times\dot{v} = (x\dot{y} - \dot{x}y)e_0$. In polar coordinates we have

$$x = r\cos\varphi \qquad \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$$
$$y = r\sin\varphi \qquad \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$$

and $|c| = x\dot{y} - \dot{x}y = r^2\dot{\varphi}$. This is the constancy of the *orbital angular momentum* of KEPLER's second law:

$$\sqrt{\ell\alpha}\, dt = r^2 d\varphi \tag{7}$$

The movement of planets on ellipses is not uniform, as r is not constant for $\varepsilon > 0$. The correct relation is the famous KEPLER equation (8) relating time and position. The proof also gives us KEPLER's third law (9).

$$\theta - \varepsilon \sin\theta = \frac{2\pi}{T}(t - t_0) \tag{8}$$

$$\frac{a^3}{T^2} = \frac{\alpha}{4\pi^2} = \frac{G(M+m)}{4\pi^2} \tag{9}$$

where T is the revolution time of the planet around the sun.

Proof. Substituting (4) in (7) $\sqrt{\ell\alpha}\,dt = a^2(1-\varepsilon\cos\theta)^2\,d\varphi = a^2\sqrt{1-\varepsilon^2}(1-\varepsilon\cos\theta)\,d\theta$ by (5). We get by integration (where t_0 corresponds to the time at perihelion):

$$\sqrt{\frac{\alpha}{a^3}}\int_{t_0}^{t} dt = \sqrt{\frac{\alpha}{a^3}}(t-t_0) = \int_0^{\theta}(1-\varepsilon\cos\theta)\,d\theta = \theta - \varepsilon\sin\theta$$

After one revolution $\theta = 2\pi$ and $t - t_0 = T$, completing the proof. □

2. ORBITAL ELEMENTS

The orbital plane of the Earth is called ecliptic, the position of the sun will be the origin and the x-axis points to the vernal point ♈. I disregard the inclination of the other planets against the ecliptic.

For our purpose the approximate positions in STANDISH [4] are sufficient. The calculations of the elements, in particular for the *eccentric anomaly* θ is done in the program `solar.c`, see appendix A.

The orbits are drawn as rotated ellipses with METAPOST with the sun in the origin. The values for the *true anomaly* are taken from the output of the `solar` program. The following visualisation of the orbits of the inner planets at their location on October 13, 2013 is generated by the METAPOST program:

```
% ------------------------------
%    Abbildung 5 -- inner planets
% ------------------------------

beginfig(5);
def orbit(expr ra,rb,an)=
    (fullcircle xscaled 2ra yscaled 2rb shifted -(ra+-+rb,0) rotated an)
enddef;
path vernal,part;
part = ((-0.25,0.25){2up+left}..(-0.2,0.5){right}.. (0,0){down});
vernal = part & reverse part reflectedabout ((0,0),up);
v=.4mm;
picture sun,mercury,venus,earth,mars;

draw fullcircle scaled 10v;
fill fullcircle scaled 2v;
sun = currentpicture;
clearit;

draw fullcircle scaled 5v shifted (0,.5v);
draw halfcircle scaled 5v rotated 180 shifted (0,5.5v);
draw (-2v,-4v)--(2v,-4v);
draw (0,-2v)--(0,-6v);
mercury = currentpicture;
clearit;
```

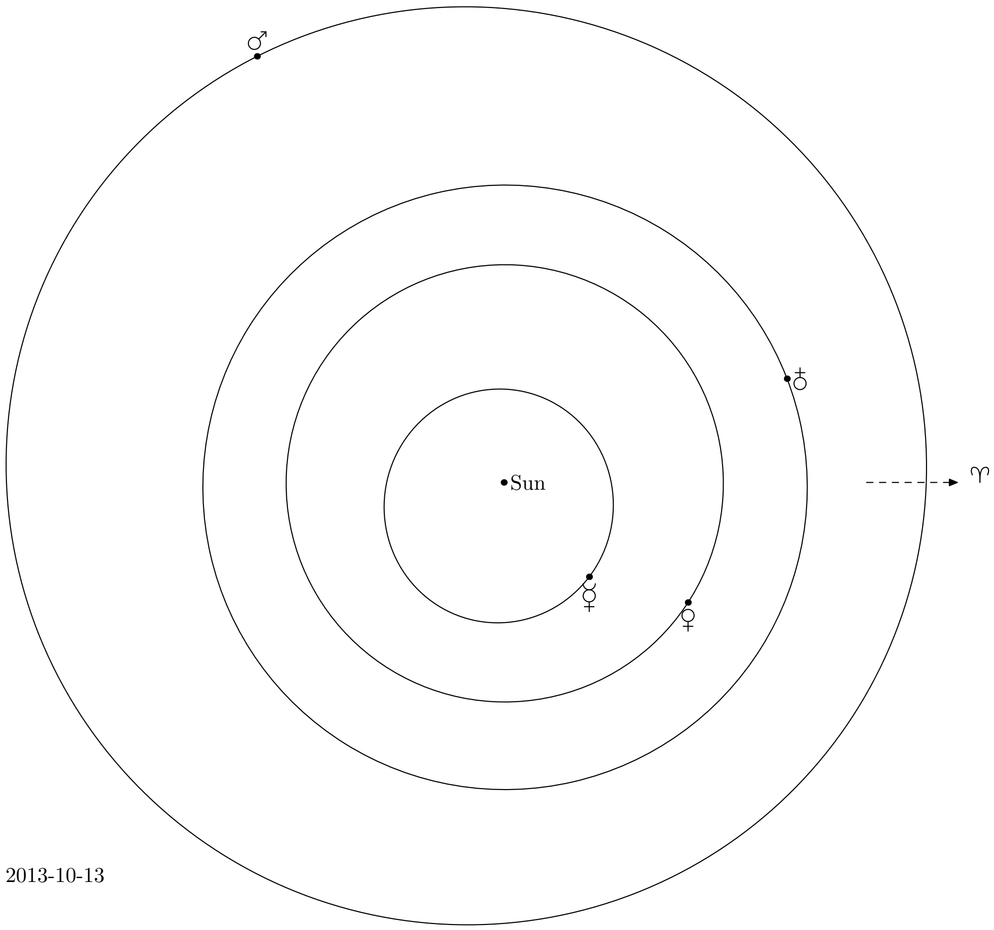
```
draw fullcircle scaled 5v shifted (0,.5v);
draw (-2v,-4v)--(2v,-4v);
draw (0,-2v)--(0,-6v);
venus = currentpicture;
clearit;

draw fullcircle scaled 5v shifted (0,-2.5v);
draw (-2v,2v)--(2v,2v);
draw (0,0)--(0,4v);
earth = currentpicture;
clearit;

path q[];
draw fullcircle scaled 5v shifted (0,-2.5v);
z10=(2.5/sqrt(2)*v,2.5/sqrt(2)*v);
z20=z10+(3v,3v);
q0=z10--z20;
q1=(x20-2v,y20)--z20;
q2=(x20,y20-2v)--z20;
draw q0 shifted (0,-2.5v);
draw q1 shifted (0,-2.5v);
draw q2 shifted (0,-2.5v);
mars = currentpicture;
clearit;

path p[];
z0=(0,0);                               % Sun
a1=3.871u; b1=0.9786*a1;                % Mercury
a2=7.233u; b2=0.99998*a2;               % Venus
a3=10u; b3=0.99986*a3;                  % Earth
a4=15.237u; b4=0.996*a4;                % Mars
% *** Output of solar 2013-10-13: ***
%julian : 2456579
%Mercury: -47.89, Perihel: 77.48
%Venus  : -33.10, Perihel: 131.78
%Earth  : 20.12, Perihel: 102.97
%Mars   : 120.08, Perihel: -23.86
label.lft(btex 2013-10-13 etex,(-13u,-13u));
v1=-47.89; v2=-33.10; v3=20.12; v4=120.08;
w1=77.48; w2=131.78; w3= 102.97; w4=-23.86;
for i=1 upto 4:
   p[i]=orbit(a[i],b[i],w[i]);
   draw p[i];
endfor
for i=1 upto 4:
   z[i]=2a[i]*dir v[i];
   p[4+i]=z0--z[i];
   z[4+i]=p[i] intersectionpoint p[4+i];
endfor
dotlabel.bot(mercury,z5);
dotlabel.bot(venus,z6);
dotlabel.rt(earth,z7);
dotlabel.top(mars,z8);
draw sun;
z9=(15u,0);
drawarrow .8z9--z9 dashed evenly;    % direction to vernal point
```

```
draw vernal scaled u shifted 1.05z9;
endfig;
%  --------------------------------------------------------------------
```



APPENDIX A. THE PROGRAM `solar.c`

The program `solar.c` calculates the true anomaly of the inner planets from a date. The calculation of the *julian* day number has been taken from the article *On calendar formulas* [2]. The routine `theta` solves the KEPLER equation (8).

```
//

/* ---------------------------------------------------------------------*

   Module:      solar.c

   Description:

        Calculating the true anomaly of inner planets
        For details: http://berndt-schwerdtfeger.de/sol/solar.pdf

        Input:  yyyy-mm-dd  (-3000 <= yyyy <= +3000)

        Output: direction of planets Mercury, Venus, Earth, Mars
```

```
   Subroutines: theta, leap, qu

   --------------------------------------------------------------------

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   limitations under the License.

   --------------------------------------------------------------------*/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>

// set constants - Keplerian elements from Standish

char *planet[] = {"Mercury", "Venus  ", "Earth  ", "Mars   "};
int em[13] = {0,0,31,59,90,120,151,181,212,243,273,304,334};
double e[4] = {0.20563661, 0.00676399, 0.01673163, 0.09336511};
double de[4]= {0.00002123,-0.00005107,-0.00003661, 0.00009149};
double L[4] = {252.25166724, 181.97970850, 100.46691572, -4.56813164};
double dL[4]= {149472.67486623, 58517.81560260, 35999.37306329, 19140.29934243};
double w[4] = {77.45771895, 131.76755713, 102.93005885, -23.91744784};
double dw[4]= { 0.15940013,   0.05679648,   0.31795260,   0.45223625};
#define PI 3.14159265358979324
#define DEG (180.0/PI)
#define RAD (PI/180.0)

// prototype statements for functions

double theta(double,double);         // theta - eps*sin theta = m
int  leap(long);                     // = 1 if year is a leap year
long qu(int,int);                    // = [a/b] Gaussklammer

// -----------------------------------------------------------------------
// main program
// -----------------------------------------------------------------------

int main(int argc, char *argv[]){

   if (argc == 1 || *argv[1] == '?'){

      printf("\nsolar v1.0, 2010-05-03 \n");
      printf("-- Copyright (C) 2010 Berndt E. Schwerdtfeger -- \n");
      printf("\nTrue anomaly of Mercury, Venus, Earth and Mars \n");
      printf("  Input:      yyyy-mm-dd  (yyyy in range -3000 - 3000) \n");
```

```
      return EXIT_SUCCESS;              // end the program
   }                                    // argc > 1 here

   int  i,d,m,y;
   long j,x;
   double T, c[4], f[4], l[4], t[4], v[4];

   sscanf(argv[1],"%d-%d-%d",&y,&m,&d);

   if (argc > 2) {
      printf("Too many parameters !\n");
      return EXIT_FAILURE;              // end the program
   }
   if (y<-3000 || y>3000) {
      printf("Error: outside allowed range -3000 <= year <= +3000 !\n\n");
      return EXIT_FAILURE;              // end the program
   }
   if (y==1582 && m==10 && d>4 && d<15) {
      printf("This date is invalid, it did never exist !\n");
      return EXIT_FAILURE;              // end the program
   }

   if (y < 1582 || y == 1582 && m*100+d < 1005) // Julian era
      j = 1721423 + d + em[m] + (y-1)*365 + qu(y-1,4);
   else {                               // Gregorian era
      x = y - 2001;
      j = 2451910 + d + em[m] + x*365 + qu(x,4) - qu(x,100) + qu(x,400);
   }
   if (m>2) j+=leap(y);                 // correction for leap year

   x = j - 2451545;                     // days since 2000-01-01
   T = (double)x/(double)36525;         // julian centuries

   for (i=0;i<4;i++) {
        c[i] = e[i]+de[i]*T;            // eccentricity
        l[i] = L[i]+dL[i]*T;            // mean longitude
        v[i] = w[i]+dw[i]*T;            // longitude of perihelion
        f[i] =(l[i] - v[i])*RAD;        // mean anomaly
        t[i] = theta(f[i],c[i]);        // eccentric anomaly
        f[i] = 2*atan(sqrt((1+c[i])/(1-c[i]))*tan(t[i]/2))*DEG+v[i]; // true a.
   }

   printf("julian : %ld\n",j);
   for (i=0;i<4;i++) printf("%s: %.2f, Perihel: %.2f\n", planet[i], f[i],v[i]);

   return EXIT_SUCCESS;                 // exit the program
}

// ---------------------------------------------------------------------
// subroutine: theta, leap, qu
// ---------------------------------------------------------------------

double theta(double m, double e){       // solve Kepler equation
   double t, d, dt;

   while (m > PI) m-=2*PI ;             // normalize m ..
   while (m <-PI) m+=2*PI ;             // .. to range -PI <= m <= PI
```

```
   t = m + e*sin(m);

   do
   {  d = m - (t - e*sin(t));
      dt= d/(1-e*cos(t));
      t += dt;
   } while (fabs(dt) > 0.0001);

   return t;
}

// ---------------------------------------------------------------------
int leap(long y){

   int i = 0;
   if (y%4 == 0)
      i = 1;
   if (y > 1582 && y%100 == 0 && y%400 != 0)
      i = 0;
   return i;
}

// ---------------------------------------------------------------------
long qu(int a, int b){

   long x = a/b;
   if (a < 0)                           // for negative numerator ..
      x -= (a%b != 0);                  // .. if remainder, subtract 1
   return x;
}
//
```

References

[1] Harold Scott MacDonald Coxeter, *Introduction to Geometry*, 2nd ed., John Wiley & Sons, 1969.

[2] Berndt E. Schwerdtfeger, *On calendar formulas* (2010), available at http://berndt-schwerdtfeger.de/wp-content/uploads/pdf/calj.pdf.

[3] ______, *Invariants of Curves of second order* (2013), available at http://berndt-schwerdtfeger.de/wp-content/uploads/pdf/c2.pdf.

[4] Erland Myles Standish, *Keplerian Elements for Approximate Positions of the Major Planets*, available at http://ssd.jpl.nasa.gov/txt/aprx_pos_planets.pdf.