REPRESENTATIONS OF COMPACT GROUPS

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Abstract. This note presents some useful every day concepts and formulas in the representation theory of compact groups, including functorial aspects like variance of the group and induction of representations.

A special section deals with representations of profinite groups.

Preface

This paper is on *linear* representations of *compact* groups in *complex finite dimensional* vector spaces. It consists of two sections, a general and a special one.

The general section on "representations and their morphisms" introduces to the category of finite dimensional representations on compact groups and their *simple* constituents, the *irreducible* representations. It also describes some functorial aspects, like variance of the group and induction of representations.

The special section on "profinite groups" applies the general theory to the situation of *profinite* groups. These are encountered as Galois groups and as $\mathfrak{p}$–adic Lie groups. Some types of representations of $SL(2, \mathfrak{o}_\mathfrak{p})$ are investigated as an example.

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1. Representations and their morphisms

1.1. Irreducible representations and Schur's Lemma. For a *compact* group K a *complex representation* is a map

$$\sigma : K \longrightarrow GL(V)$$

that is *continuous* and V is a finite dimensional complex vector space. At times it may be useful to consider other base fields than $\mathbb{C}$, but we will neglect it here. The dimension of V is called the *degree* of the representation: $\deg \sigma = \dim V$.

By transport of structure σ gives rise to a *dual* representation $\sigma^\vee : K \to GL(V^\vee)$ on the dual space: $\sigma^\vee(x) = \sigma(x^{-1})^\vee$.[1] We have $\langle \sigma^\vee(x)\alpha, \sigma(x)v\rangle = \langle \alpha, v\rangle$, or $\langle \sigma^\vee(x)\alpha, v\rangle = \langle \alpha, \sigma(x^{-1})v\rangle$, by definition. Similarly, we get representations $\sigma^\vee \otimes \tau$ on $V^\vee \otimes_\mathbb{C} W$ and $\mathrm{Hom}(\sigma, \tau)$ on $\mathrm{Hom}_\mathbb{C}(V, W)$.

The representation σ has *matrix coefficients* $\sigma_{ij} : K \to \mathbb{C}$ given in terms of a basis $(e_i)_i$ of V by $\sigma(x)e_j = \sum_i \sigma_{ij}(x)e_i$. If $(\alpha_i)_i$ is the dual basis in $V^\vee$ with $\langle \alpha_i, e_j\rangle = \delta_{ij}$ then $\sigma_{ij}(x) = \langle \alpha_i, \sigma(x)e_j\rangle$.

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[1] For the linear algebra conventions see the appendix A.

More generally, for $\alpha \in V^\vee, v \in V$ the *generalized matrix coefficient* is

$$\sigma_{\alpha v}(x) = \langle \alpha, \sigma(x)v \rangle$$

With this notation $\sigma_{\alpha_i e_j} = \sigma_{ij}$. The corresponding coefficient under the identification $V^\vee \otimes V \simeq \mathrm{End}(V)$ is $\sigma_h(x) = \mathrm{tr}\,(\sigma(x) \circ h)$ for $h \in \mathrm{End}(V)$ (see lemma A.1 in appendix A).

Remark. The matrix coefficient of the dual representation is $\sigma^\vee_{v\alpha}(x) = \sigma_{\alpha v}(x^{-1})$, so that with the usual notation $\check{\varphi}(x) = \varphi(x^{-1})$ for functions φ on K, we have $\sigma^\vee_{v\alpha} = \check{\sigma}_{\alpha v}$. We also define $\varphi^*(x) = \overline{\varphi(x^{-1})}$ for complex valued functions.

The space of *K–invariants* is defined by $V^K = \{v \in V \mid \sigma(x)v = v \ \forall x \in K\}$. We obtain K–invariants by integrating the vector valued function $x \mapsto \sigma(x)v$ for a vector $v \in V$:

$$v^\natural = \int_K \sigma(x) v dx \in V^K$$

Here dx is the Haar measure on K with $\int_K dx = 1$.

Having defined the objects, we will define the morphisms now. Let $\sigma : K \to GL(V)$, $\tau : K \to GL(W)$ be two representations. A *K–morphism* $h : V \to W$ is a linear map respecting the K–operation, i.e. for all $x \in K$: $\tau(x) \circ h = h \circ \sigma(x)$

$$\begin{array}{ccc} V & \xrightarrow{h} & W \\ {\scriptstyle \sigma(x)}\downarrow & & \downarrow{\scriptstyle \tau(x)} \\ V & \xrightarrow[h]{} & W \end{array}$$

The space of these morphisms is denoted $\mathrm{Hom}_K(V, W)$. It is the subspace of K–invariants for $\mathrm{Hom}(\sigma, \tau)$: $\mathrm{Hom}_K(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)^K$.

If h is an isomorphism, σ and τ are called *equivalent*, denoted $\sigma \simeq \tau$. They have the same matrix coefficients: if $w = h(v)$ and $\alpha = h^\vee(\beta)$ then $\sigma_{\alpha v} = \tau_{\beta w}$.

Under $V^\vee \otimes_{\mathbb{C}} W \simeq \mathrm{Hom}_{\mathbb{C}}(V, W)$ we see that $\sigma^\vee \otimes \tau \simeq \mathrm{Hom}(\sigma, \tau)$ are equivalent representations (the identification is compatible with the canonical operations).

A subspace $W \subset V$ is *stable* under K if $\sigma(x)W \subset W$ for all $x \in K$ and thus gives rise to a *sub-representation*. It is clear that there always exists a K–morphism projecting V to W: if $p : V \to W$ is any linear projector, i.e. such that $p(V) = W$ and $p(w) = w$ for $w \in W$, then $p^\natural = \int_K \sigma^\vee(x) \otimes \sigma(x) p dx \in \mathrm{Hom}_K(V, W)$ will do.

A representation is called *irreducible* if $V \neq 0$ and there are no proper stable subspaces (other than 0 and V). By the preceding remark it is clear that any representation can be decomposed into a direct sum of irreducible representations.

We just defined the *category of representations* $\mathcal{R}(K)$ of K. As kernel and image of a K–morphism $h : V \to W$ are K–stable, they are kernel and image in $\mathcal{R}(K)$. It is an abelian category. The functor $V \mapsto V^K$ is exact.

An almost obvious fact for two irreducible representations is the following

Theorem 1.1 (Schur's Lemma)**.** *Let $\sigma : K \to GL(V)$, $\tau : K \to GL(W)$ be two irreducible representations. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_K(V, W) \leq 1$$

with equality if and only if σ and τ are equivalent. Or put otherwise:

in case $\sigma = \tau$	$\mathrm{End}_K(V) = \mathbb{C}$ *and* $\mathrm{Aut}_K(V) = \mathbb{C}^\times$
in case $\sigma \not\simeq \tau$	$\mathrm{Hom}_K(V, W) = 0$

Proof. Let $h \in \mathrm{Hom}_K(V,W)$ and consider the kernel and the image: $\mathrm{Ker}\, h \subset V$ is K–stable, as well as $\mathrm{Im}\, h \subset W$.

Now by assumption of irreducibility we have two possibilities:

$$\mathrm{Ker}\, h = 0 \qquad \text{or} \qquad \mathrm{Ker}\, h = V$$

and similarly for $\mathrm{Im}\, h$. When $\mathrm{Ker}\, h = V$ or $\mathrm{Im}\, h = 0$, then $h = 0$. If $h \neq 0$ then $\mathrm{Ker}\, h = 0$ and $\mathrm{Im}\, h = W$, that is $V \simeq W$.

So either $h = 0$ or $V \simeq W$.

Let $h \in \mathrm{End}_K(V)$ and $c \in \mathbb{C}$ be an eigenvalue. Again $\mathrm{Ker}(h-c)$ is K–stable and $\neq 0$, therefore $= V$ and $h = c$. If $h \in \mathrm{Aut}_K(V)$ then $c \neq 0$ and $\mathrm{Aut}_K(V) = \mathbb{C}^\times$. □

Corollary 1.2. *If $V = \oplus_i V_i^{n_i}$ is a decomposition into irreducible representations with multiplicities then $\mathrm{End}_K(V) = \oplus_i \mathrm{End}_K(V_i^{n_i})$ and $\mathrm{End}_K(V_i^{n_i}) = M_{n_i}(\mathbb{C})$ is the ring of $n_i \times n_i$ matrices.*

In particular, $\mathrm{End}_K(V)$ is commutative if, and only if, all multiplicities are $n_i = 1$. In this case $\dim_{\mathbb{C}} \mathrm{End}_K(V)$ is the number of irreducible representations contained in V.

Let two representations be given

$$\sigma : K \longrightarrow GL(V), \quad \tau : K \longrightarrow GL(W)$$

and consider the representation $\sigma^\vee \otimes \tau$ on $V^\vee \otimes W = \mathrm{Hom}_{\mathbb{C}}(V,W)$.

Fix $\alpha \in V^\vee, w \in W$, we are going to apply Schur's Lemma to $h = (\alpha \otimes w)^\natural$:

$$h = \int_K \sigma^\vee \otimes \tau(x)(\alpha \otimes w) dx \in \mathrm{Hom}_K(V,W)$$

Applied to a vector $v \in V$ we get

$$h(v) = \int_K \sigma^\vee(x)\alpha \otimes \tau(x) w dx\; v = \int_K \langle \alpha \circ \sigma(x^{-1}), v\rangle \tau(x) w dx = \int_K \sigma_{\alpha v}(x^{-1})\tau(x) w dx$$

If σ, τ are irreducible and $\sigma \not\simeq \tau$, by Schur's Lemma $h = 0$. In the case $\sigma = \tau$ the endomorphism h is a homothety by a $c \in \mathbb{C}$ and $\mathrm{tr}\ h = d \cdot c$, where $d = \deg \sigma$.

On the other hand, the trace of h is known:

$$\mathrm{tr}\ h = \mathrm{tr}\ (\alpha \otimes w) = \langle \alpha, w\rangle$$

as $\mathrm{tr}\ (\sigma^\vee(x)\alpha \otimes \sigma(x) w) = \langle \sigma^\vee(x)\alpha, \sigma(x) w\rangle = \langle \alpha, w\rangle$ and the volume of K is 1.

We summarize this in the following

Proposition 1.3 (Schur's orthogonality relations)**.** *For irreducible σ, τ as above and $v \in V$, $\alpha \in V^\vee$, $w \in W$, $\beta \in W^\vee$ we have cases*

Case $\sigma \not\simeq \tau$:

$$\int_K \sigma_{\alpha v}(x^{-1})\tau(x) w dx = 0 \tag{1}$$

$$\int_K \sigma_{\alpha v}(x^{-1})\tau_{\beta w}(x) dx = 0 \tag{2}$$

$$\sigma_{\alpha v} * \tau_{\beta w} = 0 \tag{3}$$

Case $\sigma = \tau$*:*

$$\int_K \sigma_{\alpha v}(x^{-1})\sigma(x)w dx = \frac{\langle \alpha, w\rangle}{\deg \sigma} \cdot v \tag{4}$$

$$\int_K \sigma_{\alpha v}(x^{-1})\sigma_{\beta w}(x)dx = \frac{1}{\deg \sigma}\langle \alpha, w\rangle\langle \beta, v\rangle \tag{5}$$

$$\sigma_{\alpha v} * \sigma_{\beta w} = \frac{1}{\deg \sigma}\langle \beta, v\rangle \sigma_{\alpha w} \tag{6}$$

Proof. We have already shown (1) and (4). Applying β to them gives the relations (2) and (5). Replacing w by $\tau(y)w$ we obtain the formulas (3) and (6). $\square$

1.2. **Square integrable representations.** The group K operates on any space of functions v on K on the left and on the right: We define the so called *regular* operations by

$$\lambda(x)v(y) = v(x^{-1}y) \qquad \text{left regular representation}$$
$$\rho(x)v(y) = v(yx) \qquad \text{right regular representation}$$

We have

$$\lambda(x)(v * w) = (\lambda(x)v) * w$$
$$\rho(x)(v * w) = v * (\rho(x)w)$$

We are in particular interested in the spaces $C(K)$ and $L^2(K)$ of *continuous* resp. *square integrable* complex functions on K.

Remark. These spaces are infinite dimensional if K is not a finite group. $L^2(K)$ is a Hilbert space and ρ resp. λ are unitary representations.

We are going to describe the representation in terms of the set of equivalence classes of irreducible representations:

$$\Omega(K) = \{\omega \mid \omega = [\sigma] \text{ for an irreducible representation } \sigma : K \to GL(V)\}$$

The space of matrix coefficients $\sigma_{\alpha v}$ of an irreducible representation σ of K depends only upon the equivalence class $\omega = [\sigma]$ and will be denoted $C_\omega(K) \subset C(K)$.

Proposition 1.4. *The space* $C_\omega(K)$ *has dimension* $(\deg \omega)^2$*.*

Proof. Consider for $\alpha \in V^\vee$ the subspace $C^\alpha_\omega(K) = \{\sigma_{\alpha v} \mid v \in V\}$ and the map

$$V \longrightarrow C^\alpha_\omega(K), \quad v \longmapsto \sigma_{\alpha v}$$

We have $\rho(x)\sigma_{\alpha v} = \sigma_{\alpha,\sigma(x)v}$ and for $\alpha \neq 0$ the right hand space does not vanish. As σ is irreducible, this is an isomorphism and $C^\alpha_\omega(K)$ is an equivalent representation. In a basis $\alpha_1, \dots, \alpha_d$ of $V^\vee$ we can write $C_\omega(K) = C^{\alpha_1}_\omega(K) \oplus \cdots \oplus C^{\alpha_d}_\omega(K)$ $\square$

Remark. We have $\lambda(x)\sigma_{\alpha v} = \sigma_{\sigma^\vee(x)\alpha, v}$ and we see that the left regular representation corresponds to the dual representation.

The trace $\chi_\sigma(x) = \chi_\omega(x) = \operatorname{tr} \sigma(x) \in \mathbb{C}$ depends only on the class. It is called the *character* of the representation σ or of its class ω.

Specializing $\beta = e_i^\vee, w = e_j$ for bases $(e_i)_i$ in V resp. $(e_i^\vee)_i$ in $V^\vee$ in the orthogonality relations, we obtain $\sigma_{\alpha v} * \sigma_{e_i^\vee e_j} = \frac{1}{\deg \omega}\langle e_i^\vee, v\rangle \sigma_{\alpha e_j}$, and by summing up

$$\chi_\omega * \sigma_{\alpha v} = \sigma_{\alpha v} * \chi_\omega = \frac{1}{\deg \omega}\sum_i \langle e_i^\vee, v\rangle \sigma_{\alpha e_i} = \frac{1}{\deg \omega}\sigma_{\alpha v}$$

that is, the function $\varepsilon_\omega = \deg\omega \cdot \chi_\omega$ fixes the matrix coefficients under convolution.

For a second, inequivalent irreducible representation τ we also get

$$\sigma_{\alpha v} * \chi_\tau = 0 \quad \text{and, in particular} \quad \chi_\sigma * \chi_\tau = 0$$

A complex linear representation is *unitary* with respect to a suitable hermitian form: take any $\beta : V \times V \longrightarrow \mathbb{C}$ and remark that $\beta^\natural$ is non-singular hermitian again. By SCHUR's Lemma it is basically unique on each irreducible component.

Let $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}$ be such a non-singular K–invariant hermitian form, with semi-linearity in the left argument $\langle cu, v\rangle = \bar{c}\langle u, v\rangle$. Since $\langle \sigma(x)u, \sigma(x)v\rangle = \langle u, v\rangle$, the representation is *unitary* $\sigma : K \longrightarrow \mathbf{U}(V) \subset GL(V)$. For a vector $u \in V$ let u' be the linear form $u'(v) = \langle u, v\rangle$, then the matrix coefficients satisfy $\overline{\sigma_{u'v}(x)} = \sigma_{v'u}(x^{-1})$ and we obtain the unitary version of SCHUR's orthogonality relations:

Corollary 1.5 (unitary orthogonality relations)**.** *Let two irreducible unitary representations be given*

$$\sigma : K \longrightarrow GL(V), \quad \tau : K \longrightarrow GL(P)$$

and let $u, v \in V$ and $p, q \in P$ and consider the cases:

Case $\sigma \not\simeq \tau$:

$$\int_K \overline{\sigma_{u'v}(x)}\tau(x)q dx = 0 \tag{7}$$

$$\int_K \overline{\sigma_{u'v}(x)}\tau_{p'q}(x) dx = 0 \tag{8}$$

Case $\sigma = \tau$:

$$\int_K \overline{\sigma_{u'v}(x)}\sigma(x)q dx = \frac{\langle v, q\rangle}{\deg\sigma} \cdot u \tag{9}$$

$$\int_K \overline{\sigma_{u'v}(x)}\sigma_{p'q}(x) dx = \frac{1}{\deg\sigma}\overline{\langle u, p\rangle}\langle v, q\rangle \tag{10}$$

Theorem 1.6 (PETER–WEYL)**.** *The space $L^2(K)$ is the direct* HILBERT *sum of the finite dimensional spaces $C_\omega(K)$ spanned by the matrix coefficients of a representation of class ω, the projector onto $C_\omega(K)$ is given by convolution with ε_ω.*

For any $v \in L^2(K)$ the series $\sum_\omega v * \varepsilon_\omega$ converges to v.

Furthermore, $C_\omega(K)$ is the isotypical component in $C(K)$ of type ω.

Proof. The orthogonality of the $C_\omega(K)$ has already been seen (8). I follow WEIL [6, §§21–22].

Claim.

$$\left(\bigoplus_\omega C_\omega(K)\right)^\perp = 0$$

Proof. Assume the contrary and take $f \in \left(\bigoplus_\omega C_\omega(K)\right)^\perp, f \neq 0$. Consider $\varphi = f^* * f$ and the operator

$$T_\varphi : L^2(K) \longrightarrow L^2(K)$$
$$g \longmapsto \varphi * g$$

We first note that

$$\varphi(1) = \int_K \overline{f(x)} f(x) dx > 0$$

and therefore

$$T_\varphi\varphi(1) = \int_K \varphi(x^{-1})\varphi(x)dx = \int_K \overline{\varphi(x)}\varphi(x)dx > 0$$

since

$$\begin{aligned}\varphi(t^{-1}) &= \int_K f^*(x)f(x^{-1}t^{-1})dx = \int_K \overline{f(x^{-1})}f(x^{-1}t^{-1})dx = \\ &= \int_K \overline{f(x^{-1}t)}f(x^{-1})dx = \overline{\int_K \overline{f(x^{-1})}f(x^{-1}t)dx} = \overline{\varphi(t)}\end{aligned}$$

Hence $T_\varphi \neq 0$ and it is self–adjoint

$$\begin{aligned}\langle T_\varphi g, h\rangle &= \int_K \overline{\varphi * g(x)}h(x)dx = \iint_K \overline{\varphi(t)g(t^{-1}x)}h(x)dxdt = \\ &= \iint_K \overline{g(tx)}\varphi(t)h(x)dxdt = \iint_K \overline{g(x)}\varphi(t)h(t^{-1}x)dtdx = \\ &= \int_K \overline{g(x)}\varphi * h(x)dx = \langle g, T_\varphi h\rangle\end{aligned}$$

For each $x \in K$ the linear map $L^2(K) \longrightarrow \mathbb{C}$, $g \longmapsto \varphi * g(x)$ is continuous as $\|T_\varphi g\|_\infty \leq \|\varphi\|_2 \cdot \|g\|_2$, hence $\exists\ k_x \in L^2(K)$ with $\langle k_x, g\rangle = \varphi * g(x)$ and $\|k_x\|_2 \leq \|\varphi\|_2$ so that $k(x,y) = k_x(y)$ satisfies $\iint_K |k(x,y)|^2 dxdy \leq \int_K \|k_x\|_2^2 dx \leq \|\varphi\|_2^2$. Therefore T_φ is of Hilbert–Schmidt type, in particular *compact*. Let W be an eigenspace of a non-vanishing eigenvalue of T_φ, we have $\dim_{\mathbb{C}} W < \infty$.

As $\rho(x)T_\varphi = T_\varphi\rho(x)$, we have $\rho(x)W \subset W$ and this defines a representation of K in W. We will calculate the effect of T_φ on a matrix coefficient:

$$\begin{aligned}(T_\varphi\rho_{gh})(x) &= \int_K \varphi(t)\rho_{gh}(t^{-1}x)dt = \iint_K \overline{f(s)}f(st)\rho_{gh}(t^{-1}x)dsdt = \\ &= \iint_K \overline{f(s)}f(t)\rho_{gh}(t^{-1}sx)dsdt = \iint_K \overline{f(s)}f(t)\langle\rho(t)g, \rho(sx)h\rangle dtds = \\ &= \int_K \overline{f(s)}\langle g_1, \rho(s)h_1\rangle ds = \langle f, \rho_{g_1h_1}\rangle = 0\end{aligned}$$

by assumption, for $g_1(u) = \int_K \overline{f(t)}g(ut)dt$ and $h_1(u) = h(ux)$.

Now, any $w \in W$ is sum of matrix coefficients: write $\rho(x)w = \sum_i a_ie_i$ in an orthonormal basis, then $a_i = \rho_{e_iw}(x)$ and $w(x) = \sum_i e_i(1)\rho_{e_iw}(x)$. Therefore $T_\varphi\rho_{gh} = 0$ implies $T_\varphi w = 0$, that is $W = 0$, contradiction. □

Hence by our claim $L^2(K) = \widehat{\bigoplus}_{\omega\in\Omega(K)} C_\omega(K)$ is the spectral decomposition of the square integrable representation into finite dimensional ones. Obviously $C_\omega(K) \subset \varepsilon_\omega * L^2(K)$, but $\varepsilon_\omega * C_{\omega'}(K) = 0$ for all $\omega' \neq \omega$, hence $\varepsilon_\omega * L^2(K) = C_\omega(K)$ is the isotypical component of class ω in $L^2(K)$. □

1.3. **Induced representations.** Here I will closely follow Serre [4, §7]. Let $H \subset K$ be a *closed* subgroup and let $\sigma : K \longrightarrow GL(V)$ be a representation of K. The restriction to H gives rise to a representation of H of the same degree. The restriction is denoted by $\mathrm{Res}_H^K(\sigma)$. It is functorial $\mathrm{Res}_H^K : \mathcal{R}(K) \longrightarrow \mathcal{R}(H)$.

Now, let $H \subset K$ be an *open* subgroup (i.e. of *finite* index). Let $\tau : H \longrightarrow GL(W)$ be a representation of the subgroup. In the right regular representation of K on the space of continuous functions $C(K, W)$ we consider those functions that transform on the left under H like τ:

$$C_\tau(K, W) = \{\varphi : K \to W \mid \varphi(t\cdot x) = \tau(t)\varphi(x) \text{ for all } t \in H, x \in K\}$$

This subspace is called the *induced* representation of τ: $\rho = \mathrm{Ind}_H^K(\tau)$. Its degree is $\deg\rho = (K:H)\cdot\deg\tau$. It is functorial $\mathrm{Ind}_H^K : \mathcal{R}(H) \longrightarrow \mathcal{R}(K)$.

We have canonical H-morphisms

$$\begin{aligned}\varepsilon : W &\longrightarrow C_\tau(K,W) \\ w &\longmapsto \varepsilon_w\end{aligned} \qquad\qquad \begin{aligned}\pi : C_\tau(K,W) &\longrightarrow W \\ \varphi &\longmapsto \varphi(1)\end{aligned}$$

where

$$\varepsilon_w(x) = \begin{cases} \tau(x)w & \text{for } x \in H \\ 0 & \text{otherwise}\end{cases}$$

and with $\pi\circ\varepsilon = id$, which satisfy the

Theorem 1.7 (Frobenius reciprocity)**.** *For $\sigma : K \to GL(V)$, $\tau : H \to GL(W)$ the functors Res_H^K and Ind_H^K are adjoint to each other, in particular they are exact.*

$$\begin{aligned}\mathrm{Hom}_K(\sigma, \mathrm{Ind}_H^K(\tau)) &\xrightarrow{\sim} \mathrm{Hom}_H(\mathrm{Res}_H^K(\sigma), \tau) \\ \phi &\longmapsto \pi\circ\phi\end{aligned} \tag{11}$$

$$\begin{aligned}\mathrm{Hom}_K(\mathrm{Ind}_H^K(\tau), \sigma) &\xrightarrow{\sim} \mathrm{Hom}_H(\tau, \mathrm{Res}_H^K(\sigma)) \\ \phi &\longmapsto \phi\circ\varepsilon\end{aligned} \tag{12}$$

Proof. For (11): let $f : V \to W$ be an H–morphism of the form $f = \pi\circ\phi$, then $\phi(\sigma(x)v) = \rho(x)\phi(v)$, hence $\phi(\sigma(x)v)(1) = (\pi\circ\phi)(\sigma(x)v) = f(\sigma(x)v) = \phi(v)(x)$. Conversely, if we define $\phi : V \to C_\tau(K,W)$ by the formula $\phi(v)(x) = f(\sigma(x)v)$ we get the inverse mapping.

For (12): let $f : W \to V$ be an H–morphism of the form $f = \phi\circ\varepsilon$, then $f(w) = \phi(\varepsilon_w)$ and $\phi(\rho(x)\varepsilon_w) = \sigma(x)f(w)$. Since the support $\mathrm{supp}\,\rho(x)\varepsilon_w \subset Hx^{-1}$, the $\rho(x)\varepsilon_w$ form a basis of $C_\tau(K,W)$ when x runs through a set of representatives for K/H (a finite set) and w through a basis of W (see Serre [4, §7.1, prop. 13]). ☐

It is now clear that $\mathrm{Ind}_H^K(W)^\vee \simeq \mathrm{Ind}_H^K(W^\vee)$.

Recap some standard notation for automorphisms on K operating on various objects. If $t : K \to K$ is an automorphism we set $x^t = t^{-1}(x)$ and for a function φ on K we let ${}^t\varphi(x) = \varphi(x^t)$. If in particular $t \in K$ and the automorphism is inner conjugation this becomes the usual $x^t = t^{-1}\cdot x\cdot t$ and ${}^t\varphi(x) = \varphi(t^{-1}xt)$.

We look now at the restriction of an induction. Let $H, L < K$ be open subgroups, $\tau : H \to GL(W)$ a representation of H and let $E = \mathrm{Ind}_H^K(W)$ be its induction to K. Let $S \subset K$ be a set of representatives of the double classes $L\backslash K/H$, such that $K = \bigcup_{s\in S} L.s.H$ is a disjoint union. For $s \in S$ look at $L_s = sHs^{-1}\cap L$, subgroup of L, and consider $\tau_s = {}^s\tau|L_s$:

$$\tau_s(x) = \tau(s^{-1}xs) \qquad \text{for } x \in L_s$$

which defines a *twisted* representation $\tau_s : L_s \longrightarrow GL(W)$, also denoted by W_s. We induce the twisted W_s to representations $\mathrm{Ind}_{L_s}^L(W_s)$ of L. Now, the restriction to L of the induced representation from H to K is the direct sum of these induced twisted representations:

Proposition 1.8.

$$\mathrm{Res}_L^K\,\mathrm{Ind}_H^K(\tau) = \bigoplus_{s\in S}\mathrm{Ind}_{L_s}^L(\tau_s) \tag{13}$$

Proof. The induced representation $E = \{e : K \to W \mid e(tx) = \tau(t)e(x)\}$ decomposes $E = \bigoplus_{s\in S} E(s)$ into L–stable subspaces $E(s) = \rho(LsH)W$, which consist of those functions vanishing outside $Hs^{-1}L$: $E(s) = \{e \in E \mid \operatorname{supp} e \subset Hs^{-1}L\}$. It remains to be seen that $E(s) \simeq \operatorname{Ind}_{L_s}^L(\tau_s)$. Define

$$\begin{aligned} E(s) &\longrightarrow \operatorname{Ind}_{L_s}^L(W_s) \\ e &\longmapsto f \end{aligned}$$

by $f(x) = e(s^{-1}x)$ for $x \in L$. The reverse $e(t\cdot s^{-1}\cdot x) = \tau(t)f(x)$ for $t \in H$, $x \in L$, is easily seen to be well defined (cf. Serre [4, §7.4, prop. 15]). □

Applying (11) and (12) to the induced module $E = \operatorname{Ind}_H^K(W)$ results in

$$\operatorname{End}_K(E) \simeq \bigoplus_{s\in S} \operatorname{Hom}_H(W, E(s)) \simeq \bigoplus_{s\in S} \operatorname{Hom}_{H_s}(\operatorname{Res}_{H_s} W, W_s) \tag{14}$$

As a corollary we get the *irreducibility criterium of* Mackey

Corollary 1.9. *For the induced representation* $\operatorname{Ind}_H^K W$ *to be irreducible it is necessary and sufficient that*

(1) W *is irreducible,*
(2) $\forall\ s \in K - H$ *the two representations* $\operatorname{Res}_{H_s} W$ *and* W_s *of* H_s *are disjoint.*

Proof. Evident by (14). □

There is a different description of the endomorphism ring of an induced representation (see Casselman — *citation needed*).

Let $H, L \subset K$ be two open subgroups and $\sigma : L \longrightarrow GL(V)$ and $\tau : H \longrightarrow GL(W)$ be representations. Let

$$\mathcal{H}(\sigma,\tau) = \{h : K \to \operatorname{Hom}(V,W) \mid h(txs) = \tau(t)h(x)\sigma(s) \text{ for } s \in L, t \in H, x \in K\}$$

be the so called Hecke module. For $\sigma = \tau$ this is an algebra under convolution, the Hecke *algebra* $\mathcal{H}(\sigma)$. The dimension is $\dim_{\mathbb{C}} \mathcal{H}(\sigma,\tau) \le \operatorname{card}(H\backslash K/L) \deg\sigma \deg\tau$.

For $h \in \mathcal{H}(\sigma,\tau)$, $e \in \operatorname{Ind}_L^K(\sigma)$ the convolution

$$h * e(x) = \int_K h(xy^{-1})e(y)dy$$

is defined and is in $\operatorname{Ind}_H^K(\tau)$.

Proposition 1.10.

$$\begin{array}{ccc} \mathcal{H}(\sigma,\tau) & \xrightarrow{\sim} & \operatorname{Hom}_K(\operatorname{Ind}_L^K(\sigma), \operatorname{Ind}_H^K(\tau)) \\ h & \longmapsto & (e \mapsto h * e) \end{array} \tag{15}$$

Proof. As $\rho(x)(h * e) = h * \rho(x)e$, the convolution by h is a K–morphism. The inverse map is given as follows: let $\varphi : \operatorname{Ind}_L^K(\sigma) \longrightarrow \operatorname{Ind}_H^K(\tau)$ be a K–morphism, we define $h_\varphi(x)v = (K : L)\varphi(e_v)(x)$ for $x \in K$, $v \in V$, and it is straightforward to see this lies in $\mathcal{H}(\sigma,\tau)$.

Let h be given and $\varphi = h*$, we calculate

$$h_\varphi(x)v = (K : L)\varphi(e_v)(x) = (K : L)\int_K h(xy^{-1})e_v(y)dy$$

the function e_v being concentrated on L and $ds = (K : L)dy|L$, hence

$$h_\varphi(x)v = \int_L h(xs^{-1})\sigma(s)vds = h(x)v$$

and we get h back again.

In the other direction starting with a φ, we will check $h_\varphi * e_v$:

$$h_\varphi * e_v(x) = \int_K h_\varphi(xy^{-1})e_v(y)dy = \frac{1}{(K:L)}\int_L h_\varphi(xs^{-1})\sigma(s)vds =$$
$$= \int_L \varphi(e_{\sigma(s)v})(xs^{-1})ds = \varphi(e_v)(x)$$

hence $h_\varphi * e_v = \varphi(e_v)$, therefore $h_\varphi * e = \varphi(e)$ for all $e \in \mathrm{Ind}_L^K(\sigma)$. □

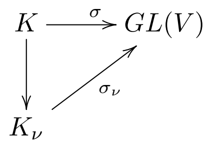
In particular we get the endomorphism ring of an induced representation as the HECKE algebra $\mathcal{H}(\sigma) = \mathrm{End}_K(\mathrm{Ind}_L^K(\sigma))$.

2. Profinite groups

A *profinite* group K is a projective limit $K = \varprojlim_\nu K_\nu$ of finite groups K_ν, each endowed with the discrete topology. It is therefore compact and totally disconnected. Conversely, a compact and totally disconnected group is profinite, as the canonical map $K \to \varprojlim K/N$ is an isomorphism, where N runs through the open normal subgroups (see SERRE [5, chap. I, §1, prop. 0]).

Examples of profinite groups are GALOIS groups and compact $\mathfrak{p}$–adic LIE groups.

Now, the representations $\sigma : K \to GL(V)$ are continuous maps into $GL(V)$, which does not contain small subgroups (see HELGASON [1, ch.II, B.5]). As the open normal subgroups $N \subset K$ form a filter base for the neighbourhoods of 1, there is some N, whose image will be trivial $\sigma(N) = \{1\}$, thus the representation factors through the finite quotient K/N. This signifies that the representations of profinite groups are composed by the canonical projection followed by a representation of a finite group:



This remark facilitates the determination of the representations of profinite groups, assuming the representations of the finite quotients to be easier to determine.

2.1. Representations of $SL(2, \mathfrak{o}_\mathfrak{p})$.

This section is based on a manuscript [2] prepared when writing up my thesis [3]. Eventually, I did not have to make use of ramified representations and the results remained unpublished.

We will see that it is quite instructive to apply the general theory of the previous section to the special situation $K = SL(2, \mathfrak{o}_\mathfrak{p})$, where $\mathfrak{o} = \mathfrak{o}_\mathfrak{p}$ is a complete discrete valuation ring and $\mathfrak{p} \subset \mathfrak{o}$ is the maximal ideal. By Theorem 1.6 (PETER–WEYL) all representations occur in $L^2(K)$ and it suffices to study $C(K) = \oplus_\omega C_\omega(K)$. To this end we will introduce filtrations by several subspaces constructed from corresponding subgroups of K.

We use the following subgroups of $K = SL(2,)$: the congruence subgroups

$$K(n) = K(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \mod \mathfrak{p}^n \right\}$$

which form a filter base for the neighbourhoods of 1 in K. Each representation σ of K is trivial on some $K(n)$, the *conductor* of σ is $\mathfrak{p}^n$, if n is the smallest value such that $K(n) \subset \mathrm{Ker}\,\sigma$.

We also consider the HECKE groups

$$K_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \mod \mathfrak{p}^n \right\}$$

and

$$K_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a-1 \equiv d-1 \equiv c \equiv 0 \mod \mathfrak{p}^n \right\}$$

We note the intersection $U = \bigcap_n K_1(n)$ and $B = \bigcap_n K_0(n)$ (Borel subgroup) of (upper) triangular matrices.

With $j = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ the transpose of a matrix is ${}^t x = j x^{-1} j^{-1}$.

Let $V(n) = {}^{K(n)}C(K) = C(K(n)\backslash K)$, $V_1(n) = {}^{K_1(n)}C(K) = C(K_1(n)\backslash K)$, $V_0(n) = {}^{K_0(n)}C(K) = C(K_0(n)\backslash K)$.

To the pattern of subgroups

$$\begin{array}{ccccccccccccc}
K(0) & \supset & K(1) & \supset & K(2) & \supset & \dots & \supset & K(n) & \supset & \dots & \supset & 0 \\
\| & & \cap & & \cap & & \dots & & \cap & & \dots & & \cap \\
K_1(0) & \supset & K_1(1) & \supset & K_1(2) & \supset & \dots & \supset & K_1(n) & \supset & \dots & \supset & U \\
\| & & \cap & & \cap & & \dots & & \cap & & \dots & & \cap \\
K_0(0) & \supset & K_0(1) & \supset & K_0(2) & \supset & \dots & \supset & K_0(n) & \supset & \dots & \supset & B
\end{array}$$

corresponds the pattern of representations

$$\begin{array}{ccccccccccccc}
\mathbb{C} = V(0) & \subset & V(1) & \subset & V(2) & \subset & \dots & \subset & V(n) & \subset & \dots & \subset & C(K, \mathbb{C}) \\
\| & & \cup & & \cup & & \dots & & \cup & & \dots & & \cup \\
\mathbb{C} = V_1(0) & \subset & V_1(1) & \subset & V_1(2) & \subset & \dots & \subset & V_1(n) & \subset & \dots & \subset & C(U\backslash K, \mathbb{C}) \\
\| & & \| & & \cup & & \dots & & \cup & & \dots & & \cup \\
\mathbb{C} = V_0(0) & \subset & V_0(1) & \subset & V_0(2) & \subset & \dots & \subset & V_0(n) & \subset & \dots & \subset & C(B\backslash K, \mathbb{C})
\end{array}$$

In the sequel we will determine all representations in the lower line: these are exactly the representations V that contain a B invariant vector, $V^B \neq 0$, and each occurs with multiplicity 1. In characteristic $\neq 2$ each $V_0(n)$ contains $2n$ representations (for $n \geq 1$), those that occur new at level n contain the B invariant function

$$\delta_{K_0(n-1)} - q\delta_{K_0(n)} \pm \sqrt{(-1)^{\frac{q-1}{2}} q} \cdot \sum_{t \in \mathbb{F}_q^\times} \left(\frac{t}{q}\right) \delta_{K_0(n) \begin{pmatrix} 1 & 0 \\ t\pi^{n-1} & 1 \end{pmatrix}}$$

where $\pi \in \mathfrak{p}$ is a local uniformising element and for a subset $A \subset K$ the function δ_A is defined by $\delta_A(x) = 1$ if $x \in A$ and $\delta_A(x) = 0$ otherwise.

The space $C(U\backslash K, \mathbb{C})$ contains all the *non cuspidal* representations.

Appendix A. Linear algebra conventions

For a complex vector space V let $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the dual vector space of linear forms $\alpha : V \to \mathbb{C}$, the effect on a vector $v \in V$ will be denoted by $\langle \alpha, v \rangle = \alpha(v)$.

We make use of the identification $V^\vee \otimes_{\mathbb{C}} W \simeq \mathrm{Hom}_{\mathbb{C}}(V, W)$ as follows:

$$\begin{array}{ccc}
V^\vee \otimes_{\mathbb{C}} W & \simeq & \mathrm{Hom}_{\mathbb{C}}(V, W) \\
\alpha \otimes w & & v \mapsto \langle \alpha, v \rangle w
\end{array}$$

We identify the dual of $V^\vee$ with V: $V^{\vee\vee} = V$ and denote the value of the linear form v on $V^\vee$ by $\langle v, \alpha \rangle = v(\alpha) = \alpha(v) = \langle \alpha, v \rangle$.

Taking the dual is a contravariant equivalence in the category of vector spaces:

$$\begin{array}{ccc} \mathrm{Hom}(V,W) & \xrightarrow{\sim} & \mathrm{Hom}(W^\vee,V^\vee) \\ h & \mapsto & h^\vee \end{array}$$

defined by $h^\vee(\beta) = \beta \circ h$. The contravariance is $(h \circ g)^\vee = g^\vee \circ h^\vee$ in a situation like $U \xrightarrow{g} V \xrightarrow{h} W$.

With these identifications this gives rise to the following commutative diagram

$$\begin{array}{ccc} V^\vee \otimes W & \longrightarrow & \mathrm{Hom}(V,W) \\ \downarrow{\scriptstyle\vee} & & \downarrow{\scriptstyle\vee} \\ W \otimes V^\vee & \longrightarrow & \mathrm{Hom}(W^\vee,V^\vee) \end{array}$$

the vertical map on the left is given by $(\alpha \otimes w)^\vee = w \otimes \alpha$.

Note the useful trace formula:

Lemma A.1. *For $\alpha \in V^\vee, v \in V$ the trace of the endomorphism $\alpha \otimes v$ is*

$$V^\vee \otimes V \simeq \mathrm{End}(V) \xrightarrow{\mathrm{tr}} \mathbb{C}$$

$$\mathrm{tr}\ \alpha \otimes v = \langle \alpha, v \rangle$$

Proof. Let $(e_i)_i$ and $(e_i^\vee)_i$ be dual bases in V resp. $V^\vee$, i.e. such that $\langle e_i^\vee, e_j \rangle = \delta_{ij}$, then $v = \sum_i \langle e_i^\vee, v \rangle e_i$ and the trace of h is $\mathrm{tr}\ h = \sum_i \langle e_i^\vee, h(e_i) \rangle$. Hence we get

$$\begin{aligned} \mathrm{tr}\ (\alpha \otimes v) &= \sum_i \langle e_i^\vee, \alpha \otimes v(e_i) \rangle = \sum_i \langle e_i^\vee, \langle \alpha, e_i \rangle v \rangle = \\ &= \sum_i \langle \alpha, e_i \rangle \langle e_i^\vee, v \rangle = \sum_i \langle \alpha, \langle e_i^\vee, v \rangle e_i \rangle = \langle \alpha, v \rangle \end{aligned}$$

□

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Index