ARTIN'S L-FUNCTIONS AFTER WEIL, GROTHENDIECK

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To Jo on his $3F^{th}$ birthday.

Preface

This paper summarizes some notes taken when comparing Weil's presentation of Artin's L-functions in [4] with Grothendieck's in [1].

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1. Artin L-functions according to Weil

1.1. **Dirichlet-series.** In general we will use the notation of Weil in [5, VII] and [4, II]. We recap some basic notation.

Let k denote a global field, i.e. either an algebraic number-field (a finite field extension $k/\mathbb{Q}$) or an algebraic function-field of dimension 1 over a finite field of constants. For each place v of k its completion is denoted k_v. For finite places the local ring of integers is $\mathfrak{o}_v$ and its maximal ideal $\mathfrak{p}_v$.

The free abelian group of all $\mathfrak{p}_v$ is called group of *divisors* and is denoted $\mathfrak{M}$; it is written *multiplicatively.* A divisor $\mathfrak{m} \in \mathfrak{M}$ is *positive* if the exponents of all $\mathfrak{p}_v$ are positive. The positive divisors form the semigroup $\mathfrak{M}_+$. In characteristic 0 (number-fields) $\mathfrak{M}$ corresponds to the group of fractional ideals of k and $\mathfrak{M}_+$ to the ideals $\neq 0$ in the ring of integers $\mathfrak{o}_k$ of k. In characteristic > 1 (function-fields) $\mathfrak{M}$ corresponds to the *line bundles* on the associated complete curve – this aspect will not be delved into in this article. As we want to compare Weil's treatment of Artin's L-functions with Grothendieck's we are particularly interested in the *function-field* case.

Let $\Omega_k = \mathrm{Hom}(\mathbf{A}_k^\times/k^\times, \mathbb{C}^\times)$ be the group of (not necessarily unitary) characters on the idele class group of k. Weil attaches to a function $c : \mathfrak{M}_+ \longrightarrow \mathbb{C}$ an *extended* Dirichlet-series belonging to k by

$$L(c, \omega) = \sum_{\mathfrak{m} \in \mathfrak{M}_+} c(\mathfrak{m})\omega(\mathfrak{m})$$

This series is absolutely convergent for $\mathrm{Re}\,\omega > \alpha + 1$ if and only if $|c(\mathfrak{m})| \leq C|\mathfrak{m}|^{-\alpha}$ for some $C > 0$ and some $\alpha \in \mathfrak{R}$.

The coefficients $c(\mathfrak{m})$ are uniquely determined by the function $L(c, \omega)$.

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1.2. **Artin-Hecke L-series.** Let W_k be the Weil group of k. The quotient by its derived group W_k/W_k' is canonically isomorphic to the idele-class group $\mathbf{A}_k^\times/k^\times$ by class field theory, let $\alpha_k : \mathbf{A}_k^\times/k^\times \xrightarrow{\sim} W_k/W_k'$ denote the Artin reciprocity map. So, we can also identify the characters $\omega \in \Omega_k$ on the idele-class group with the abelian characters $\chi : W_k/W_k' \longrightarrow \mathbb{C}^\times$ (representations of degree 1).

Let X_k be the ring of representations of W_k (the Weil group). If k'/k is a separable extension of degree d, then $W_{k'}$ is of index d in W_k and we can induce representations from $W_{k'}$ to W_k, denoted $\chi' \mapsto [\chi'; k'/k]$, and considered as a mapping $X_{k'} \longrightarrow X_k$.

To each finite separable extension k'/k and to every character $\chi' \in X_{k'}$ there is attached a divisor $f_{k'}(\chi')$ of k' and an extended Dirichlet series $L_{k'}(\chi')$ satisfying the following properties ([4, §73])

(1) For $\deg \chi' = 1$ the divisor $f_{k'}(\chi') = f(\omega')$ is the conductor of $\omega' = \chi' \circ \alpha_{k'}$ and the L-series is the usual Hecke series to the Größencharakter ω' as in Weil [5, VII] $L_{k'}(\chi') = L(\omega')$.
(2) The mappings $\chi' \mapsto f_{k'}(\chi')$ and $\chi' \mapsto L_{k'}(\chi')$ are *homomorphisms* of the additive group $X_{k'}$ into the multiplicative groups of the divisors $\mathfrak{M}_{k'}$ of k', and of the Dirichlet series belonging to k' with initial coefficient 1, respectively.
(3) For $k''/k'/k$ let $D(k''/k')$ be the *discriminant* of k'' over k', let χ'' be any character of $W_{k''}$ of degree $n = \deg \chi''$. Then

$$f_{k'}([\chi''; k''/k']) = \mathrm{N}_{k''/k'}(f_{k''}(\chi''))D(k''/k')^n$$
$$L_{k'}([\chi''; k''/k']) = [L_{k''}(\chi''); k''/k']$$

From these properties it is clear that these conditions determine the L-series for all characters.

For function fields Weil has proven the Artin conjecture that these L-functions are *holomorph* for positive characters $\chi \neq 1$ ([2, II^e partie, §V, n^o 27-28]).

2. Artin L-functions according to Grothendieck

2.1. **Definition of Z- and L-functions.** In [1], following Weil [3, p. 507], Grothendieck discussed the Z- and L-functions for a scheme Y of finite type over a finite field $\mathbb{F}_q$. Let the finite group G operate on the right such that $X = Y/G$ exists, let $\rho : G \longrightarrow GL(V)$ be a representation in a finite dimensional vector space over a field F of characteristic 0 (classically $F = \mathbb{C}$, but we might also take $F = \mathbb{Q}_\ell$ or $F = \mathbb{C}_\ell$).

The Artin L-function is defined by

$$L(Y, G, X, t) = L(Y/X, \rho, t) = \prod_{x \in X^0} \frac{1}{\det(1 - \rho^\natural(x) t^{d(x)})}$$

where X^0 = set of *closed* points of X, $d(x) = \deg x = [\kappa(x) : \mathbb{F}_q]$ is the residual degree and

$$\rho^\natural(x) = \frac{1}{|T_y|} \sum_{\substack{s \in G_y \\ s \to \varphi_y}} \rho(s) \in \mathrm{End}(V)$$

where $y \to x$ for a point y in the fiber Y_x, with G_y the *decomposition* group, T_y is the *inertia* group ($e(y/x) = |T_y| = e(x)$ is independent of y/x) and $\varphi_y \in G_y/T_y \simeq G(\kappa(y)/\kappa(x))$ is the Frobenius in the Galois group of the residual extension.

We have

$$\log L(Y/X,\rho,t) = \sum_{\nu\geq 1} c_\nu(Y,G,\rho)t^\nu/\nu$$
$$c_\nu(Y,G,\rho) = \sum_{x\in X(\mathbb{F}_{q^\nu})} Tr_\rho^\natural(\varphi_x)$$

where

$$Tr_\rho^\natural(\varphi_x) = \frac{1}{|T_y|}\sum_{\substack{s\in G_y\\ s\to\varphi_x}} Tr_\rho(s)$$

where $y \in Y(\mathbb{F}_{q^\nu})$ is above the point $x \in X(\mathbb{F}_{q^\nu})$ and φ_x is the Frobenius.

References

[1] Alexander Grothendieck, *Formule de Lefschetz et rationalité des fonctions L*, Sém. Bourbaki **279** (1964/65), 1–15.

[2] André Weil, *Courbes algébriques et variétés abéliennes*, Hermann, Paris, 1948. réimp. 1971.

[3] ———, *Numbers of solutions of equations in finite fields*, Bull. Am. Math. Soc. **55** (1949), 497–508, available at http://berndt-schwerdtfeger.de/wp-content/uploads/pdf/nf.pdf.

[4] ———, *Dirichlet Series and Automorphic Forms*, Lecture Notes in Math., vol. 189, Springer, Berlin, 1971.

[5] ———, *Basic Number Theory*, Classics in Mathematics, Springer, Berlin, Heidelberg, New York, 1973, 1995.