GUIDE TO ℓ-ADIC COHOMOLOGY

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Abstract. This paper guides thru the jungle of SGA 4–5 to the major facts on ℓ-adic cohomology.

Preface

This guide (Leitfaden) was presented in the winter term 1977/78 in the seminar of Claus Michael Ringel at University of Bonn. Its purpose is to break a path thru the jungle of SGA 4–5, to gather the most important definitions and theorems of ℓ–adic cohomology. For the proofs exact signposts into SGA are given.

Section 1 contains the definition of the abelian category of $\mathbb{Z}_\ell$–sheaves, together with suitable enveloping and quotient (pro–) categories, as well as the interpretation of $\widetilde{X}_{\ell,\mathrm{smooth}}$ by ℓ–adic representations (for this we need the covering theory — in the appendix I give a more detailed account than was possible in the talks).

Section 2 contains the definition and functorial properties of the ℓ-adic cohomology: the ∂-functor, Leray spectral sequence, compatibility with arbitrary base change, long exact sequence for an open subspace and its complement, cohomological dimension, Künneth-formulas.

Mainz, October 14, 2001 B. E. Schwerdtfeger

Preface to version 1.1. The TEX v1.0 has been typed from my (German) manuscript, that had been photocopied and distributed during the seminar. But there exists an earlier version 0 of the manuscript, which contains more details, but also lots of striked thru paragraphs and tentative formulations, which did not make it into the final guide.

After looking thru the original pages again I decided to include some of the material that I have left out in v1.0.

Changes made in this version include: Fixing several typos, newly introduced into the TEX version. In particular, the equivalence of $\widetilde{X}_\ell$ with a quotient category of Artin–Rees objects, defining the $\mathcal{P}ro$ category of $\widetilde{X}_{\acute{e}t,\mathrm{cons}}$ in section 1.1, erroneously stated projective systems in $\widetilde{X}_\ell$, which made it look like a vicious circle.

I added some more remarks that enhance (I hope) the legibility and explain or motivate some of the constructions.

The Künneth–type formulas for $\mathbb{Z}_\ell$–sheaves have been explicited. In v1.0 it was simply stated that the situation is *slightly more complicated* than for $\mathbb{Q}_\ell$–sheaves – which gave you no clue of how complicated it might be. Instead of a direct sum we have a short exact sequence (in the flat case) or two short exact sequences (reminiscent of spectral sequences) in general, with pretty explicit terms controlling the torsion part - so, no surprise.

In total, I wished I had been more detailed - but after all, it was only meant as a guide to the literature. I might want to give a fuller account though, some day.

Mainz, July 10, 2002 B. E. Schwerdtfeger

2010 *Mathematics Subject Classification.* Primary 14F20; Secondary 14F30, 20C11.
Key words and phrases. ℓ-adic cohomology, ℓ-adic representation, étale fundamental group.
 version 1.1.505, March 4, 2015.

1. Category $\widetilde{X}_\ell$ of ℓ-adic sheaves on X

As to prerequisites: the short note on flat descent [6] contains enough of the definitions (and some properties in particular for the flat topology) for the categories of sheaves that are of basic concern to us in this paper.

Basically, the ℓ–adic sheaves are constructed from the finite étale group schemes via projective limits, similar to the construction of the ℓ–adics from the finite ℓ–torsion groups.

1.1. $\mathbb{Z}_\ell$–sheaves $\widetilde{X}_\ell$.

X is a noetherian scheme.

A sheaf $\mathcal{F} \in \widetilde{X}_{\acute{e}t}$ is called *constructible* (which is then written $\mathcal{F} \in \widetilde{X}_{\acute{e}t,\mathrm{cons}}$), when $\exists\, X = \bigcup X_i$ a finite union of subschemes and $\forall i$ $\mathcal{F}|X_i$ is representable by a *finite* étale group over X_i.

These are the noetherian objects in the abelian category $\widetilde{X}_{\acute{e}t}$. (see SGA 4 [5], IX 2.4 and SGA $4\frac{1}{2}$ Arcata IV 3.2 [1]).

$\widetilde{X}_\ell$ is by definition the category of projective systems $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ of constructible sheaves $\mathcal{F}_n$ with the condition

$$\mathcal{F}_{n+1}/\ell^n\mathcal{F}_{n+1} \simeq \mathcal{F}_n \tag{ℓ-adic}$$

(see SGA$4\frac{1}{2}$, Rapport 2.1 [1], SGA5[4],VI,1.1).

$\widetilde{X}_\ell$ is a full subcategory of the category of projective systems $[\mathbb{N}, \widetilde{X}_{\acute{e}t,\mathrm{cons}}]$. The $\mathbb{Z}_\ell$–sheaves even build an abelian category (see below), although you cannot calculate the kernel and cokernel componentwise (as the condition (ℓ-adic) may be violated).

There are non-zero projective systems, whose 'projective limit' vanishes. The technical term will now be considered. A sheaf $\mathcal{F}$ will be called *AR-null* (for Artin–Rees), when there exists a k such that $\mathcal{F}_{n+k} \longrightarrow \mathcal{F}_n$ is 0 for any n. These build a thick (Serre) subcategory AR_0 (SGA5 [4], 2.2).

AR-$\widetilde{X}_\ell$ will denote the category of those $\mathcal{F}$, which are AR-isomorphic to a $\mathbb{Z}_\ell$-sheaf:

$$\widetilde{X}_\ell \subset \text{AR-}\widetilde{X}_\ell \subset [\mathbb{N}, \widetilde{X}_{\acute{e}t,\mathrm{cons}}]$$

(so AR-$\widetilde{X}_\ell$ contains all objetcs giving the 'same projective limit', thus enveloping $\widetilde{X}_\ell$). Then we have an equivalence of categories

$$\widetilde{X}_\ell \xrightarrow{\sim} \text{AR-}\widetilde{X}_\ell/AR_0 \quad (\subset [\mathbb{N}, \widetilde{X}_{\acute{e}t,\mathrm{cons}}]/AR_0 = \mathcal{P}ro(\widetilde{X}_{\acute{e}t,\mathrm{cons}}))$$

The advantage of the envelope is that kernels and co-kernels can be calculated componentwise in AR-$\widetilde{X}_\ell$ (SGA5 [4], 5.2.1). The inclusion $\widetilde{X}_\ell \hookrightarrow$ AR-$\widetilde{X}_\ell$ is *not* exact.

$\mathcal{F} \in \widetilde{X}_\ell$ is *smooth* ($\mathcal{F} \in \widetilde{X}_{\ell,\mathrm{smooth}}$) – or twisted constant – when each component $\mathcal{F}_n$ is representable on X (by a finite commutative étale group). For a connected X we have

$$\begin{array}{ccc} \widetilde{X}_{\ell,\mathrm{smooth}} & \xrightarrow{\sim} & \mathbb{Z}_\ell[\pi_1(X,x)] - Mod \\ \mathcal{F} & \mapsto & M = \varprojlim_n \mathcal{F}_{n,x} \end{array} \tag{1}$$

an equivalence of categories, see the appendix A ([4, SGA 5, VI, 1.2.5]). x is a geometric point: $x \in X(\Omega)$, Ω separably closed.

Example 1.1. For $\ell \neq$ residue class characteristic is $\mathcal{F} = (\mu_{\ell^n})_n$ smooth. This abelian sheaf is also denoted by $\mathbb{Z}_\ell(1)$.

1.2. $\mathbb{Q}_\ell$–**sheaves** $\widetilde{X}_\ell \otimes \mathbb{Q}_\ell$**.** Each $\mathbb{Z}_\ell$–sheaf has a canonical module structure over the (smooth) ring sheaf $\underline{\mathbb{Z}_\ell} = (\mathbb{Z}/\ell^n)_n$. The abelian group of morphisms $\mathrm{Hom}_{\widetilde{X}_\ell}(\mathcal{F}, \mathcal{G}) = \varprojlim_n \mathrm{Hom}(\mathcal{F}_n, \mathcal{G}_n)$ is a $\mathbb{Z}_\ell$–module and will also be written $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{F}, \mathcal{G})$. The ($\ell$–) torsion sheaves in $\widetilde{X}_\ell$ constitute a thick subcategory, the quotient of which will be denoted $\widetilde{X}_\ell \otimes \mathbb{Q}_\ell$. It can also be described like this: the objects are the same as in $\widetilde{X}_\ell$, the morphisms are

$$\mathrm{Hom}_{\mathbb{Q}_\ell}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

(see SGA5 [4], VI, 1.4.2).

For a connected scheme there is again the equivalence

$$\widetilde{X}_{\ell,\mathrm{smooth}} \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \mathbb{Q}_\ell[\pi_1(X,x)] - Mod$$

the smooth $\mathbb{Q}_\ell$–sheaves correspond to the ℓ–adic representations on X

$$\rho : \pi_1(X,x) \longrightarrow \mathrm{Aut}(V) \qquad\qquad \dim_{\mathbb{Q}_\ell} V < \infty$$

In topology this corresponds to the equivalence of local systems of vector spaces (= locally constant sheaves of complex vector spaces) with *discrete* vector bundles on X, i.e. sets with π_1- operation.

In [8], Serre constructs to linear representations of certain algebraic groups over number fields ($X = \mathrm{Spec}\, K$) $\ell-$adic representations for every ℓ (loc.cit. II, 2.5).

2. ℓ-ADIC COHOMOLOGY WITH OR W/O COMPACT SUPPORT

2.1. **Definition of ℓ-adic cohomology.** Let $f : Y \longrightarrow X$ be a separated morphism of finite type. The pull back induces an exact functor

$$f^* : \text{AR-}\widetilde{X}_\ell \longrightarrow \text{AR-}\widetilde{Y}_\ell$$

defined component wise, which throws AR_0 into AR_0. Therefore we get an exact functor

$$f^* : \widetilde{X}_\ell \longrightarrow \widetilde{Y}_\ell$$

The (higher) direct images with compact support induce

$$R^q_! f : \widetilde{Y}_{\acute{e}t,cons} \longrightarrow \widetilde{X}_{\acute{e}t,cons}$$

[5, XVII,5.3.6], [1, Arcata IV 6.2] and thereby applying component wise [4, V,5.3.1] (theorem of Shih)

$$\text{AR-}\widetilde{Y}_\ell \longrightarrow \text{AR-}\widetilde{X}_\ell$$

which again applies AR_0 into AR_0, which finally results in

$$R^q_! f : \widetilde{Y}_\ell \longrightarrow \widetilde{X}_\ell$$

(here we can no longer calculate component wise).

Analogously, the normal direct images ('without support') of sheaves with torsion prime to the residue characteristics induce

$$R^q f : \widetilde{Y}_{\acute{e}t,cons,prim} \longrightarrow \widetilde{X}_{\acute{e}t,cons}$$

[1, Th.finitude 1.1]. Hence for $\ell \neq$ residue characteristics first a component wise map $\text{AR-}\widetilde{Y}_\ell \longrightarrow \text{AR-}\widetilde{X}_\ell$, and finally

$$R^q f : \widetilde{Y}_\ell \longrightarrow \widetilde{X}_\ell$$

2.2. **Properties of ℓ-adic cohomology.**

2.2.1. *Long exact sequence.*

$$R^q_! f, R^q f : \widetilde{Y}_\ell \longrightarrow \widetilde{X}_\ell$$

are *exact ∂-functors*: When

$$0 \to \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \to 0$$

is exact in $\widetilde{Y}_\ell$, then we have the exact sequence in $\widetilde{X}_\ell$

$$\ldots \longrightarrow R^q_! f(\mathcal{F}') \longrightarrow R^q_! f(\mathcal{F}) \longrightarrow R^q_! f(\mathcal{F}'') \longrightarrow R^{q+1}_! f(\mathcal{F}') \longrightarrow \ldots$$

and simlilar for the cohomology without support.

If f is compactifiable: $f : Y \hookrightarrow \overline{Y} \xrightarrow{\overline{f}} X$ with j open immersion, $\overline{f}$ proper, then we have $R^q_! f = R^q \overline{f} \circ j_!$. In particular for a proper f: $R^q_! f = R^q f$.

Remark. The $R^q f$ $(q \geq 1)$ are effaceable, therefore the derived of $f_* = R^0 f$ in the sense of homological algebra. For the $R^q_! f$ this is *wrong* in general ($j_!$ is exact, but does not in general map injectives into injectives). A simple counter example: Let X/k be a complete smooth curve over the algebraically closed field k, $U \subsetneqq X, U \neq \varnothing$ (an affine curve), $f : U \longrightarrow \operatorname{Spec} k$. We have $R^0_! f = f_! = \overline{f}_* \circ j_! = 0$, and therefore $R^2 f_! = 0$, but $R^2_! f(\mu_n) = \mathbb{Z}/n$ [1, Arcata VI,2.1].

2.2.2. *Leray spectral sequence.* For $Z \xrightarrow{g} Y \xrightarrow{f} X$ we have

$$R^p_! f \circ R^q_! g \Rightarrow R^{p+q}_! (f \circ g)$$
$$R^p f \circ R^q g \Rightarrow R^{p+q} (f \circ g)$$

[4, SGA 5, VI, 2.2.4]

2.2.3. *Compatibility with base change (only for compact support).*

$$\begin{tikzcd} Y & Y' \arrow[l, "\varphi'"'] \arrow[d, "f'"] \\ X \arrow[u, leftarrow, "f"] & X' \arrow[l, "\varphi"] \end{tikzcd}$$

where $Y' = Y \underset{X}{\times} X'$, and f of finite type and separated, φ arbitrary, let $\mathcal{F} \in \widetilde{Y}_\ell$.

$$\varphi^* R^q_! f(\mathcal{F}) \simeq R^q_! f'(\varphi'^* \mathcal{F})$$

as isomorphism of ∂–functors $\widetilde{Y}_\ell \longrightarrow \widetilde{X}'_\ell$.

[4, SGA 5,VI,2.2.3 B], [5, SGA 4, XVII, 5.2.6].

2.2.4. *Relative Cohomology for an open subspace and its Complement.* Let $j : V \to Y$ be *open*, $i : Z = Y - V \to Y$ the closed complement. Let $f : Y \longrightarrow X$ be *compactifiable*, $f_V = f|V = f \circ j$, $f_Z = f|Z = f \circ i$, $\mathcal{F} \in \widetilde{Y}_\ell$. Then we have the following long exact sequence in $\widetilde{X}_\ell$:

$$\cdots \to R^q_! f_V(\mathcal{F}|V) \longrightarrow R^q_! f(\mathcal{F}) \longrightarrow R^q_! f_Z(\mathcal{F}|Z) \longrightarrow R^{q+1}_! f_V(\mathcal{F}|V) \to \ldots$$

[4, SGA 5, VI, 2.2.3 C], [5, SGA 4, XVII, 5.1.16.2]

Remark. I am not aware of a proof without the assumption ‘compactifiable’.

Remark. In topology the term $R^q_! f_V(\mathcal{F}|V)$ corresponds to $H^q(X, Z; \mathcal{F})$, the cohomology of X mod Z ([2, théorème 4.10.1]). There is an analogous sequence for $R^q f$:

$$\cdots \to R^q_Z f(\mathcal{F}) \longrightarrow R^q f(\mathcal{F}) \longrightarrow R^q f_Z(\mathcal{F}|Z) \longrightarrow R^{q+1}_Z f(\mathcal{F}) \to \dots$$

where $R^q_Z f(\mathcal{F}) = R^q(f_{Z*} \circ i^!)(\mathcal{F})$ is a *relative* cohomology (classically: cohomology classes with (arbitrary) support $\subset Z$).

2.2.5. *Vanishing of higher Cohomology.* Let the fiber dimension be $\leq n$, then we have

$$q > 2n \quad \Longrightarrow \quad R^q_! f(\mathcal{F}) = 0$$

All occuring functors are additive in $\mathcal{F}$, hence torsion ℓ–sheaves are mapped to torsion sheaves and all assertions remain valid for $\mathbb{Q}_\ell$–sheaves.

2.3. **The case of a base field k.** We consider $f : X \longrightarrow S = \operatorname{Spec} k$, k a field, let $\Gamma_k = \pi_1(S, s)$ be the Galois group of $\bar{k}/k$, $s : k \hookrightarrow \bar{k}$.

By (1) above we have $\widetilde{S}_\ell \xrightarrow{\sim} \mathbb{Z}_\ell[\Gamma_k] - Mod$ ($\mathbb{Z}_\ell$–sheaves are smooth over a field). Hence the direct image of cohomology modules become:

$$\begin{aligned} H^q_!(\overline{X},\ \ ) &: \widetilde{X}_\ell \longrightarrow \mathbb{Z}_\ell[\Gamma_k] - Mod \\ H^q_!(\overline{X}, \mathcal{F}) &= \varprojlim_n R^q_! f(\mathcal{F}_n)_s \qquad \text{by base change} \\ &= \varprojlim_n H^q_!(\overline{X}_{\acute{e}t}, \overline{\mathcal{F}}_n) \end{aligned}$$

where $\overline{X} = X \otimes_k \bar{k}$. (It was allowed to compute componentwise, as AR-Null objects vanish in $\varprojlim$).

Γ_k operates by structure transport.

2.4. **Künnethformulas.** [5, SGA 4, XVII, 5.4.3], [9, Verdier].

To the formulas in the literature, which are formulated in the derived category $D(X, \mathbb{Z}/\ell^n)$, we apply the cohomology functor and the functor "$\varprojlim$" and get the following:

Let $f' : Y' \longrightarrow X'$, $f'' : Y'' \longrightarrow X''$ be as above (separated of finite type) over a scheme S.

Let

$$f = f' \underset{S}{\times} f'' : Y = Y' \underset{S}{\times} Y'' \longrightarrow X = X' \underset{S}{\times} X''$$

Let $\mathcal{F}' \in \widetilde{Y}'_\ell \otimes \mathbb{Q}_\ell$, $\mathcal{F}'' \in \widetilde{Y}''_\ell \otimes \mathbb{Q}_\ell$ be $\mathbb{Q}_\ell$–sheaves. Let $\mathcal{F} = \mathcal{F}' \boxtimes_{\mathbb{Q}_\ell} \mathcal{F}''$ be their *exterior* tensor product (i.e. pull back to Y and there its $\otimes_{\mathbb{Q}_\ell}$–product). Then we have

$$R^n_! f(\mathcal{F}) = \bigoplus_{p+q=n} R^p_! f'(\mathcal{F}') \boxtimes_{\mathbb{Q}_\ell} R^q_! f''(\mathcal{F}'')$$

For $\mathbb{Z}_\ell$–sheaves the situation is slightly more complicated ([9]), as follows.

If at least one of the two sheaves is *locally free* (i.e. a *vector bundle*), then you get an exact sequence of the following form

$$0 \to \bigoplus_{p+q=n} R^p_! f'(\mathcal{F}') \boxtimes_{\mathbb{Z}_\ell} R^q_! f''(\mathcal{F}'') \to R^n_! f(\mathcal{F}) \to \bigoplus_{p+q=n+1} \mathcal{T}or_1^{\mathbb{Z}_\ell}(R^p_! f'(\mathcal{F}'), R^q_! f''(\mathcal{F}'')) \to 0$$

Without any flatness assumption you only obtain two sequences of the following kind instead (resulting from corresponding spectral sequences):

$$0 \to \bigoplus_{p+q=n} R^p_! f'(\mathcal{F}')\boxtimes_{\mathbb{Z}_\ell} R^q_! f''(\mathcal{F}'') \to \mathcal{H}^n \to \bigoplus_{p+q=n+1} \mathcal{T}or_1^{\mathbb{Z}_\ell}(R^p_! f'(\mathcal{F}'), R^q_! f''(\mathcal{F}'')) \to 0$$

and

$$0 \to R^{n+1}_! f(\mathcal{T}or_1^{\mathbb{Z}_\ell}(\mathcal{F}', \mathcal{F}'')) \to \mathcal{H}^n \to R^n_! f(\mathcal{F}' \boxtimes_{\mathbb{Z}_\ell} \mathcal{F}'') \to 0$$

So, you get could control of the deviation from the flat case.

Appendix A. Fundamental group and covering theory

This section owes a lot to the presentation that I have given in the Topological Covering Theory [7], which basically goes back to courses given by Giraud and Verdier. In particular the last section on Galois coverings could be taken over almost as is.

Let Ω be a separably closed field. We restrict ourselves to the unramified theory.

Lemma A.1. *In the commutative diagram*

$$\begin{tikzcd} Y \arrow[rr, shift left, "\alpha"] \arrow[rr, shift right, "\beta"'] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

let f be unramified and separated, and Y be connected.

If there is a field K and a rational point $y \in Y(K)$ with $\alpha(y) = \beta(y)$, then we have $\alpha = \beta$.

Proof. Consider the coincidence scheme $Y_{\alpha,\beta} \subset Y$ of α, β:

$$\begin{tikzcd} Y_{\alpha,\beta} \arrow[r, hook] \arrow[d] & Y \arrow[d, "\gamma"] \\ X \arrow[r, hook, "\delta"'] & X \times_S X \end{tikzcd}$$

where γ is the composed map of the diagonal embedding followed by the fiber product map $\alpha \underset{S}{\times} \beta$:

$$\gamma : Y \hookrightarrow Y \times_S Y \xrightarrow{\alpha \underset{S}{\times} \beta} X \times_S X$$

The condition says that $Y_{\alpha,\beta} \neq \varnothing$ (as $y \in Y_{\alpha,\beta}(K)$). Now f is unramified $\Leftrightarrow$ δ is open immersion and f is separated $\Leftrightarrow$ δ is closed immersion.

Therefore $Y_{\alpha,\beta} \neq \varnothing$, open and closed in the connected Y, which implies $Y_{\alpha,\beta} = Y$ and the Lemma is proved. □

Lemma A.2. *Let S be connected.*

$$\begin{tikzcd} Y \arrow[rr, "\alpha"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

with f, g finite and étale. Assume there is a geometric point $s \in S(\Omega)$, such that the fibers over s under α are bijective: $\alpha : g^{-1}(s) \xrightarrow{\sim} f^{-1}(s)$.

Then α is an isomorphism.

Proof.
$$Y = \operatorname{Spec} \mathcal{B} \longrightarrow \operatorname{Spec} \mathcal{A}$$
with locally free $\mathcal{O}_S$–algebras $\alpha^* : \mathcal{A} \longrightarrow \mathcal{B}$ of finite rank. As S is connected, the rank is constant, say n resp. m. Now we have $f^{-1}(s) = \operatorname{Hom}_{\kappa(s)}(\mathcal{A}(s), \Omega)$, $\mathcal{A}(s) \otimes_{\kappa(s)} \Omega \simeq \Omega^n$, therfor $n = \# f^{-1}(s) = \# g^{-1}(s) = m$.

If we would have $\alpha(Y) \neq X$, then $f(X - \alpha(Y)) = S$, as $X - \alpha(Y)$ is open and closed, therefore also the image under f.

Thus there would exist a $x \in (X - \alpha(Y))(\Omega)$ with $f(x) = s$, which is in contradiction to the surjectivity of α on the fiber over s.

Hence $\alpha(Y) = X$ and $\forall\, x \in X(\Omega)$ we have $\# \alpha^{-1}(x) \geq 1$.

Now α is finite and étale and the degree has to be 1 (as $n = m$). But then α is an isomorphism, qed. □

Let $f : X \to S$ be finite, étale, and X, S both connected and $\neq \varnothing$.

By Lemma A.2 we have $\operatorname{Hom}_S(X, X) = \operatorname{End}_S(X) = \operatorname{Aut}(X/S)$. The opposite group $G = G(X/S) = \operatorname{Aut}(X/S)^{opp}$ operates on the right of X/S.

It is equivalent:

$$\exists\, x \in X(\Omega) \quad x \cdot G = f^{-1}(f(x)) \tag{2}$$

$$\forall\, x \in X(\Omega) \quad x \cdot G = f^{-1}(f(x)) \tag{3}$$

$$\begin{aligned} X \times G &\xrightarrow{\sim} X \underset{S}{\times} X \\ (x, \sigma) &\longmapsto (x, x\sigma) \end{aligned} \tag{4}$$

Proof. (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) is obvious. If (2) holds, apply Lemma A.2 to
$$e : X \times G \xrightarrow{\sim} X \underset{S}{\times} X, \qquad e(x, \sigma) = (x, x\sigma)$$
□

Remark. Under these conditions X/S is also called a *principal bundle* (with *structure group* G) or a *torsor*.

Definition A.1. X/S is called a *Galois covering* with group $G(X/S)$, iff these conditions are met.

By Lemma A.1 $G(X/S)$ operates freely
$$G \xrightarrow{\sim} G \cdot x \subset f^{-1}(f(x))$$
and *Galois* signifies *transitively*.

Lemma A.3. *Let X/S be Galois, Y/S be étale covering.*

Then $G(X/S)$ operates simply transitive on $\operatorname{Hom}_S(Y, X)$

Proof. Without restriction let $\operatorname{Hom}_S(Y, X) \neq \varnothing$, let $\alpha : Y \to X/S$, let $y \in Y(\Omega)$ and consider $x = \alpha(y)$, $f(x) = s$, $f : X \to S$
$$\begin{array}{ccccc} G(X/S) & \longrightarrow & \operatorname{Hom}_S(Y, X) & \longrightarrow & f^{-1}(s) \\ \rho & \longmapsto & \rho \circ \alpha, \;\; \beta & \longmapsto & \beta(y) \end{array}$$
By Lemma A.1 they are injective, by Galois they are surjective, qed. □

Theorem A.4. *Let X/S be a connected étale covering $\neq \varnothing$. Then there exists a Galois covering Y/S with the property $P(Y)$*
$$Y \times \operatorname{Hom}_S(Y, X) \xrightarrow{\sim} Y \underset{S}{\times} X \qquad /Y$$
and any Z/S with property $P(Z)$ factors thru Y/S.

Proof. Let $s \in S(\Omega)$ and $f^{-1}(s) = \{x_1, \dots, x_n\} \subset X(\Omega)$, $n = \deg f$. Let

$$Y \subset (X/S)^n = X \underset{S}{\times} \dots \underset{S}{\times} X$$

be the connected component of $(x_1, \dots, x_n)$ (or more accurately: of the images of $(x_1, \dots, x_n) \in (X/S)^n(\Omega)$ in $(X/S)^n$).

By Lemma A.1 we have for

$$\begin{aligned} e : Y \times \mathrm{Hom}_S(Y, X) &\longrightarrow Y \underset{S}{\times} X \qquad /Y \\ (y, \alpha) &\longmapsto (y, \alpha(y)) \end{aligned}$$

(defined for T–valued points) that $e(K)$ is injective for all fields K. Each i defines $\alpha_i : Y \subset (X/S)^n \longrightarrow X$ and $\alpha_i(x_1, \dots, x_n) = x_i$, the fiber over $(x_1, \dots, x_n)$ is mapped surjectively, therefore e is an isomorphism.

Let us show now that $g : Y \longrightarrow S$ is Galois. Let $y \in g^{-1}(s)$, as $\alpha_i(y) \in f^{-1}(s) = \{x_1, \dots, x_n\}$ there is a $\sigma \in \mathbf{S}_n$ with $\alpha_i(y) = x_{\sigma(i)}$, i.e. $y = (x_1, \dots, x_n)^\sigma$. This σ induces a $(X/S)^n \longrightarrow (X/S)^n$ and as $Y \cap \sigma(Y) \neq \varnothing$ we must have $Y = \sigma(Y)$, hence $\sigma \in G(Y/S)$.

The assertion on Z (universal property) is clear. $\square$

Theorem A.5. *In the situation $h : Z \xrightarrow{g} Y \xrightarrow{f} X$ let Z/X be Galois. Then we have Z/Y is Galois and Y/X is Galois exactly if $G(Z/Y) \lhd G(Z/X)$ is a normal divisor. Moreover $G(Z/X)$ operates transitively on $\mathrm{Hom}_X(Z, Y)$. In the Galois case we have canonically*

$$G(Y/X) \xrightarrow{\sim} \mathrm{Hom}_X(Z, Y) \xrightarrow{\sim} G(Z/X)/G(Z/Y)$$

Proof. Let $z \in Z$, $y = g(z)$, $x = f(y)$ and consider the diagram

$$\begin{tikzcd} G(Z/X) \arrow[r, "\sim"] & h^{-1}(x) & \tau \mapsto \tau(z) \\ G(Z/Y) \arrow[u, hook] \arrow[r, hook] & g^{-1}(y) \arrow[u, hook] \end{tikzcd}$$

If $\tau(z) \in g^{-1}(y)$, then we have $g \circ \tau(z) = g(z)$, hence by Lemma A.1 $g \circ \tau = g$, i.e. $\tau \in G(Z/Y)$ and therefore Z/Y is Galois. Furthermore the isotropy group of $g \in \mathrm{Hom}_X(Z, Y)$ in $G(Z/X)$ is exactly $G(Z/Y)$, so

$$G(Z/Y) \backslash G(Z/X) \hookrightarrow \mathrm{Hom}_X(Z, Y)$$

Now, the set on the right has at most $\deg f = \deg Y/X$ elements (Lemma A.1) and the set on the left has exactly $\deg h / \deg g = \deg f$ elements, which implies $\mathrm{Hom}_X(Z, Y) = \{g \circ \tau \mid \tau \in G(Z/X)\}$.

Now let us investigate the case Y/X Galois: then by Lemma A.3 $G(Z/X) \xrightarrow{\sim} \mathrm{Hom}_X(Z, Y)$, $\rho \mapsto \rho \circ g$ is bijective. This gives us a canonical mapping

$$\begin{array}{ccc} G(Z/X) & \longrightarrow & G(Y/X) \\ \tau & \longmapsto & \rho \end{array} \qquad \text{where } \rho \circ g = g \circ \tau$$

and we see immediately that this is a homomorphism. The kernel $G(Z/Y)$ is therefore a normal divisor.

Now let $G(Y/X)$ be a normal divisor and let us show that Y/X is Galois, that is $\#G(Y/X) = \deg f$. Let $\tau \in G(Z/X)$, $y \in Y$ be given. For any two $z, z' \in g^{-1}(y)$ there is $\sigma \in G(Z/Y)$ with $z' = \sigma(z)$. By assumption $\tau \sigma \tau^{-1} \in G(Z/Y)$, hence $g \circ \tau \sigma = g \circ \tau$, and $g(\tau(z')) = g(\tau(z))$ and $g \circ \tau$ is constant on the fiber, so that the definition $\rho(y) := g(\tau(z))$, for any $z \in g^{-1}(y)$ is meaningful. This shows the surjectivity of $G(Y/X) \longrightarrow \mathrm{Hom}_X(Z, Y)$. $\square$

Definition A.2 (fundamental group)**.**

$$\pi_1(X, x) := \varprojlim_{i \in I} G(X_i/X)$$

where

$$I = \{(X_i, x_i) \mid f_i : X_i \longrightarrow X \quad \text{connected étale covering, with } f_i(x_i) = x\}$$

($x \in X(\Omega), x_i \in X_i(\Omega)$ are *geometric* points).

By Lemma A.1 (again), there is from $i = (X_i, x_i)$ to $j = (X_j, x_j)$ at most *one* X–morphism – this defines an order relation $i \geq j$ on I.

The following theorem describes the equivalence of étale coverings on X with the finite sets, on which the fundamental group operates continuously.

Theorem A.6.

$$\begin{array}{ccll} X_{\acute{e}t,fin} & \xrightarrow{\sim} & \mathit{Fin}\text{–}\pi_1(X,x)\text{–}\mathit{Sets} & \\ Y/X & \longmapsto & \mathrm{Hom}_X(x,Y) = f^{-1}(x) & \qquad \textit{geometric fiber} \end{array}$$

is an equivalence of categories.

Proof. In the opposite direction: let M be a finite π_1–set, there is an open subgroup, which operates trivially, by theorem A.4 there is an i such that $\pi_1(X_i, x_i)$ operates trivially. Hence M can be considered as $\pi_1(X,x)/\pi_1(X_i,x_i) = G(X_i/X)$ set. On $X_i \times M$ operates $G_i = G(X_i/X)$ by $(t,m)\cdot\sigma = (t\cdot\sigma, \sigma^{-1}\cdot m)$ and $Y = (X_i \times M)/G_i$ is independent from the choice of i. Y/X correspondents to M under the equivalence. □

For more details rummage in SGA 1 [3, chap. V].

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