ÉTALE MORPHISMS IN TOPOLOGY

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Abstract. This paper discusses the following types of continuous maps in Topology: étale, separated, proper and covering morphisms, and investigates their relationship. The Galois theory for finite coverings is discussed in more detail.

Preface

The subject of this note are *étale morphisms* in topology, as they are encountered in the theory of *unramified coverings*. It goes back to classes of Giraud [2] and Verdier [6]. A previous version had been published earlier [5].

Berlin, December 9, 2007 B. E. Schwerdtfeger

1. Terms that play a role

We will move in the category of topological spaces. We will not define those, but assume the reader to have a basic knowledge. The neighbourhood filter of a point $x \in X$ is denoted by $\mathfrak{V}(x)$ (see Bourbaki [1]).

Definition 1.1. A morphism $f : Y \to X$ is called *étale* if

$\forall\, y \in Y\ \exists\, V \in \mathfrak{V}(y)$ with $U = f(V) \in \mathfrak{V}(f(y))$ and $f|V : V \xrightarrow{\sim} U$ is homeomorph.

Definition 1.2. A morphism $f : Y \to X$ is called *separated* if

$$\forall\, y_1 \neq y_2 \text{ with } f(y_1) = f(y_2)\ \exists\, V_1 \in \mathfrak{V}(y_1), V_2 \in \mathfrak{V}(y_2) \text{ with } V_1 \cap V_2 = \varnothing$$

Definition 1.3. A morphism $f : Y \to X$ is called *proper* if f is *closed* with *quasi–compact* fibers $f^{-1}(x) \subset Y$, $x \in X$.

Definition 1.4. A morphism $f : Y \to X$ is called a *covering* if $\forall\, x \in X$ the fiber $f^{-1}(x)$ is *discrete* and there exists a neighbourhood U of x and a homeomorphism h with

$$\begin{array}{ccc} h : f^{-1}(U) & \xrightarrow{\sim} & U \times f^{-1}(x) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle p_1} & \\ U & & \end{array}$$

where the fiber $f^{-1}(x)$ over x is mapped to $\{x\} \times f^{-1}(x)$, i.e. you can assume that $h(y) = (f(y), y) = (x, y)$ for $y \in f^{-1}(x)$.

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Otherwise put, a covering is a *locally trivial bundle* with a *discrete* fiber.

In an equivalent description you have

$$f^{-1}(U) = \bigcup_{y \in f^{-1}(x)} V_y \quad \text{disjoint, } V_y \in \mathfrak{V}(y) \;\text{ and }\; f|V_y \xrightarrow{\sim} U$$

In particular, a *covering* is *separated* and *étale.*

In the sequel we will investigate the functorial behaviour of the above classes of morphisms in these situations:

base change: given a map $f : Y \to X$ and an arbitrary base change $X_1 \xrightarrow{\varphi} X$

$$\begin{array}{ccc} Y & \longleftarrow & Y_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ X & \underset{\varphi}{\longleftarrow} & X_1 \end{array}$$

with $Y_1 = Y \underset{X}{\times} X_1$, is the property preserved for f_1 ?

composition: given two maps $f : Y \to X$ and $g : Z \to Y$, is the property preserved for $h = f \circ g : Z \to X$?

2. Étale morphisms

Proposition 2.1. *Étale maps are stable under base change:*

$$\begin{array}{ccc} Y & \overset{\psi}{\longleftarrow} & Y_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ X & \underset{\varphi}{\longleftarrow} & X_1 \end{array}$$

where $Y_1 = Y \underset{X}{\times} X_1$*. If* f *is étale, then* f_1 *is étale.*

Let $Z \xrightarrow{g} Y \xrightarrow{f} X, h = f \circ g$*. If two of the maps* f, g, h *are étale, the third is étale as well.*

Proof. Let us prove *stability under base change*: pick $y_1 \in Y_1$, let $y = \psi(y_1)$ and $x_1 = f_1(y_1)$, such that $y_1 = (y, x_1)$ with $f(y) = \varphi(x_1) = x$. As f is étale there is $V \in \mathfrak{V}(y)$ such that $U = f(V) \in \mathfrak{V}(x)$ satisfies $f|V : V \xrightarrow{\sim} U$. Set $V_1 = \psi^{-1}(V)$ and $U_1 = \varphi^{-1}(U)$ then I claim that $f_1|V_1 : V_1 \xrightarrow{\sim} U_1$. But since $V_1 = V \underset{U}{\times} U_1$ and $f_1(v, u_1) = u_1$ this is obvious.

Let us prove the second assertion: it is clear that the composition of étale maps is étale, we are in a local situation like

$$\begin{array}{ccccc} W & \xrightarrow{\sim} & V & \xrightarrow{\sim} & U \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xrightarrow{g} & Y & \xrightarrow{f} & X \end{array} \qquad h = f \circ g : Z \to X$$

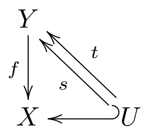
and f, g *étale* $\Longrightarrow h$ *étale*, and f, h *étale* $\Longrightarrow g$ *étale*, and g, h *étale* $\Longrightarrow f$ *étale.* $\square$

Obviously, *étale* mappings $f : Y \longrightarrow X$ are *open*, as this is a local property. But this also holds for their *sections*:

Lemma 2.2. *Let $U \subset X$ be open, $s : U \longrightarrow Y$ be a section of f, i.e. $f \circ s = id_U$. Then $V = s(U) \subset Y$ is open.*

Proof. Take a $y \in V$, we will show $V \in \mathfrak{V}(y)$. Say $y = s(x)$ for $x \in U$, hence $f(y) = x$. Let $W \in \mathfrak{V}(y)$ be such that $f|W : W \xrightarrow{\sim} f(W) \subset U$ and consider $f^{-1}s^{-1}W \cap W \in \mathfrak{V}(y)$. For $z \in W \cap f^{-1}s^{-1}W$ we have $f(s \circ f(z)) = f(z)$, which implies $s \circ f(z) = z \in s(U) = V$, therefore $W \cap f^{-1}s^{-1}W \subset V$ and V is a neighbourhood of y. □

Lemma 2.3. *Let $U \subset X$ be open, $s, t : U \longrightarrow Y$ sections with $s(x) = t(x)$ at one point $x \in U$. Then $\exists$ a neighbourhood U_1 of x such that $s|U_1 = t|U_1$.*



Proof. $s(U) \cap t(U)$ is an open neighbourhood of $s(x) = t(x)$. If $y \in s(U) \cap t(U)$, then $y = s(f(y)) = t(f(y))$, therefore $s|U_1 = t|U_1$ for $U_1 := f(s(U) \cap t(U))$. □

Corollary 2.4. *Let $f : Y \to X$ be étale, $h : Z \to X$ arbitrary, $\sigma, \tau : Z \to Y$ morphisms $/X$ (i.e. $f \circ \sigma = f \circ \tau$) with $\sigma(z) = \tau(z)$ at one point $z \in Z$. Then $\exists W \in \mathfrak{V}(z)$ such that $\sigma|W = \tau|W$.*

Proof. Define sections $s, t : Z \to Z \underset{X}{\times} Y$, $s(z) = (z, \sigma(z))$, $t(z) = (z, \tau(z))$. □

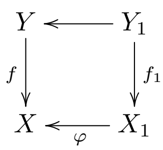
Remark. The category of *étale spaces* over X is equivalent to the category of *sheaves* on X, see Godement [3, II, §1.2] *L'espace étalé attaché à un faisceau*; this category is the basic example of a *topos* $Top(X)$, see SGA 4 [4, IV, 2.1] *Topos associé à un espace topologique*. To an *étale* mapping $f : F \longrightarrow X$ under this equivalence is associated the *sheaf of sections* $\mathcal{F}$ defined by $\mathcal{F}(U) := \Gamma(U, F) = \{s : U \to F \mid f \circ s = id_U\}$. The *fiber* over x is discrete and by Lemma 2.3 isomorphic to the *stalk* of the sheaf

$$f^{-1}(x) = F_x \xrightarrow{\sim} \mathcal{F}_x = \varinjlim_{U \in \mathfrak{V}(x)} \mathcal{F}(U)$$

by sending a $y \in F_x$ to the germ of the section $(f|V)^{-1}$, $V \in \mathfrak{V}(y)$ suitably chosen. The reverse is done by mapping the *germ* $s_x \in \mathcal{F}_x$ to the *value* $s(x) \in F_x$.

3. Separated morphisms

Proposition 3.1. *Separated maps are stable under base change:*



where $Y_1 = Y \underset{X}{\times} X_1$. If f is separated, then f_1 is separated.

In a diagram $Z \xrightarrow{g} Y \xrightarrow{f} X$: f, g separated $\Longrightarrow f \circ g$ separated $\Longrightarrow g$ separated.

Proof. f *separated* is equivalent to *the diagonal* $\Delta_Y \subset Y \underset{X}{\times} Y$ *is closed.* For the canonical map $\psi : Y_1 \underset{X_1}{\times} Y_1 \longrightarrow Y \underset{X}{\times} Y$ we have $\psi^{-1}(\Delta_Y) = \Delta_{Y_1}$. The last assertions follow from the definitions. □

Proposition 3.2. *A section s of a separated morphism*

$$f : Y \overset{s}{\underset{}{\rightleftarrows}} X$$

is a closed embedding.

Proof. For *embedding* holds for any section and *closed* follows from $s(X) = t^{-1}(\Delta_Y)$, where t is the section on Y pulled back from s: $t(y) = (s \circ f(y), y)$

$$\begin{array}{ccc} Y & \longleftarrow & Y \times_X Y \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} & & \downarrow\uparrow{\scriptstyle t} \\ X & \underset{f}{\longleftarrow} & Y \end{array}$$

□

Lemma 3.3. *Let $f : Y \to X$ be étale and separated, Z connected and $h : Z \to X$ arbitrary. Then $\forall\, z \in Z$ the maps*

$$\mathrm{Hom}_X(Z, Y) \longrightarrow f^{-1}(h(z))$$
$$\sigma \longmapsto \sigma(z)$$

are injective.

Proof. Let $\sigma, \tau \in \mathrm{Hom}_X(Z, Y)$ and define $g : Z \to Y \times_X Y$ by $g(z) := (\sigma(z), \tau(z))$. $g^{-1}(\Delta_Y)$ is closed and open (Cor. 2.4), hence $g^{-1}(\Delta_Y) = Z$ if $\neq \varnothing$, i.e. $\sigma = \tau$. □

4. Proper morphisms

Lemma 4.1. *Let $f : Y \to X$ be proper, then it is quasi–compact: $\forall\, K \subset X$ quasi–compact $\Longrightarrow f^{-1}(K) \subset Y$ is quasi–compact.*

Proof. Start with a family of open sets $(V_\alpha)_\alpha$ such that $f^{-1}(K) \subset \bigcup_\alpha V_\alpha =: V$. For any finite index subset I define $V_I := \bigcup_{\alpha \in I} V_\alpha$ and $U_I := X - f(Y - V_I)$, $U := X - f(Y - V)$. Obviously $K \subset U$, $U_I \subset U$.

Now, for $u \in U$ we have $f^{-1}(u) \subset V$, and by quasi–compactness of the fibers there exists I such that $f^{-1}(u) \subset V_I$, that is $u \in U_I$ and thus $K \subset \bigcup_I U_I$. By quasi–compactness of K we can find finitely many I, that is there is an I with $K \subset U_I$. This implies $f^{-1}(K) \subset V_I$. □

Proposition 4.2. *Proper maps are stable under base change:*

$$\begin{array}{ccc} Y & \longleftarrow & Y_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ X & \underset{\varphi}{\longleftarrow} & X_1 \end{array}$$

where $Y_1 = Y \underset{X}{\times} X_1$.

If f is proper, then f_1 is proper.

Proof. For $x_1 \in X_1$ the fiber $f_1^{-1}(x_1) = f^{-1}(\varphi(x_1)) \times \{x_1\}$ is quasi–compact.

To show that f_1 is closed let $A \subset Y_1$ be closed and let us show that $X_1 - f_1(A)$ is open. Consider a point $x_1 \in X_1 - f_1(A)$. For any $y \in f^{-1}(\varphi(x_1))$ we have $(y, x_1) \in Y_1 - A$, therefore there are neighbourhoods V of y and U_1 of x_1 with

$V \times U_1 \cap A = \varnothing$. As the fiber is quasi-compact a finite number of the V cover the fiber. Replace V with this finite union and U_1 with the corresponding finite intersection: we have found an open $V \supset f^{-1}(\varphi(x_1))$ and and $U_1 \ni x_1$ with $V \times U_1 \cap A = \varnothing$. Set $U := X - f(Y - V)$, then $\varphi(x_1) \in U$ and U is open in X, by continuity of φ and eventually restricting U_1 further we may assume $\varphi(U_1) \subset U$. This implies $f^{-1}\varphi(U_1) \subset V$ and from this we get $x_1 \in U_1 \subset X_1 - f_1(A)$. □

Proposition 4.3. *In a diagram* $Z \xrightarrow{g} Y \xrightarrow{f} X$ *we have*

(1) $$f, g \text{ are proper} \Longrightarrow f \circ g \text{ is proper}$$

(2) $$f \circ g \text{ is proper, } g \text{ surjective} \Longrightarrow f \text{ is proper}$$

(3) $$f \circ g \text{ is proper, } f \text{ separated} \Longrightarrow g \text{ is proper}$$

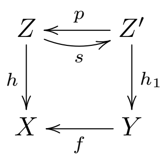
Proof. (1) is clear by the lemma 4.1.

(2) Let $h = f \circ g$.

$\forall B \subset Y$ is $f(B) = h(g^{-1}(B))$, hence f closed.

$\forall x \in X$ is $f^{-1}(x) = g(h^{-1}(x))$, hence f quasi–compact.

(3) Apply base change: $Z' = Z \underset{X}{\times} Y$, consider



h_1 is proper by base change (Prop. 4.2), the section s is defined by $s(z) := (z, g(z))$. Now, p is separated as a base change of f (Prop. 3.1), hence the section s is a closed embedding (Prop. 3.2), in particular it is proper. It follows by (1) that $g = h_1 \circ s$ is proper. □

5. Finite coverings

Definition 5.1. If all fibers of a covering $f : Y \to X$ are *finite*, then the map $X \to \mathbb{N}$, $x \mapsto \#f^{-1}(x)$ is *locally constant* on X and f is called *locally finite covering*. It is called (globally) *finite*, if all fibers have the same number n of points, which is called its *degree* : $\deg f = n = \#f^{-1}(x)$, $\forall x \in X$.

Proposition 5.1. *A separated étale morphism* $f : Y \to X$ *such that* $x \mapsto \#f^{-1}(x)$ *is locally constant, is a locally finite covering.*

Proof. Without loss of generality assume $n = \#f^{-1}(x) \quad \forall x \in X$ (restricting to such a neighbourhood). $f^{-1}(x) = \{y_1, \dots, y_n\}$. There are open neighbourhoods $V_1, \dots, V_n$ of $y_1, \dots, y_n$, pairwise disjoint, with $f|V_i$ is homeomorph to its image. Define $U := \bigcap_i f(V_i)$, then with $W_i := f^{-1}(U) \cap V_i$ we have $f(W_i) = U$ and $f^{-1}(U) = \bigcup_i W_i$ disjoint. □

Theorem 5.2. $f : Y \to X$ *is a locally finite covering if and only if* f *is étale, separated and proper.*

Proof. "⇒" It remains to show 'proper'. The fibers are finite, so they are quasi–compact. Let us show 'closed'. Obviously $X - f(Y)$ is open, hence $f(Y)$ closed (and open), so that we may assume $X = f(Y)$. Let $B \subset Y$ be closed, $x \notin f(B)$, say U a neighbourhood of x with $f^{-1}(U) = V_1 \dot\cup \cdots \dot\cup V_n$, $f|V_i : V_i \xrightarrow{\sim} U$, and since

$f^{-1}(x) \subset Y - B$ we can assume (eventually shrinking V_i) that $V_i \subset Y - B$. Hence $f^{-1}(U) \subset Y - B$, that is $U \cap f(B) = \varnothing$ and $X - f(B)$ is open.

"$\Leftarrow$" Let f be étale, separated and proper. $x \in X$, the fiber $f^{-1}(x)$ is discrete and quasi–compact, therefore finite, say $f^{-1}(x) = \{y_1, \ldots, y_n\}$.

As $X - f(Y)$ is open, we can assume $x \in f(Y)$, that is $n \geq 1$. There are pairwise disjoint open sets $W_1, \ldots, W_n$ with $y_i \in W_i$ and $f|W_i : W_i \xrightarrow{\sim} f(W_i)$. Set

$$U := f(W_1) \cap \cdots \cap f(W_n) \cap (f(Y) - f(Y - (W_1 \cup \cdots \cup W_n)))$$

U is an open neighbourhood of x. With $V_i := f^{-1}(U) \cap W_i$ is by construction

$$f^{-1}(U) = V_1 \,\dot\cup \cdots \dot\cup\, V_n \quad \text{and} \quad f|V_i : V_i \xrightarrow{\sim} U$$

□

This implies good functorial properties through the propositions 2.1, 3.1, 4.2, 4.3.

Corollary 5.3. *Stability under base change:*

$$\begin{array}{ccc} Y & \longleftarrow & Y_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ X & \longleftarrow & X_1 \end{array}$$

where $Y_1 = Y \underset{X}{\times} X_1$.

If f is locally finite covering, then f_1 is locally finite covering. A finite covering is stable under base change.

In a diagram $Z \xrightarrow{g} Y \xrightarrow{f} X, h = f \circ g$ *we have*

$$\begin{aligned} f, g \text{ are locally finite coverings } &\Longrightarrow h \text{ is locally finite covering.} \\ f, h \text{ are locally finite coverings } &\Longrightarrow g \text{ is locally finite covering.} \\ g, h \text{ are locally finite coverings with surjective } g &\Longrightarrow f \text{ is locally finite covering.} \end{aligned}$$

From the formula

$$\#h^{-1}(x) = \sum_{y \in f^{-1}(x)} \#g^{-1}(y)$$

we also deduce that

$$\begin{aligned} f, g \text{ are finite coverings } &\Longrightarrow h \text{ is a finite covering.} \\ g, h \text{ are finite coverings with surjective } g &\Longrightarrow f \text{ is a finite covering.} \end{aligned}$$

Note. g need not be finite, if f and h are finite, e.g. if Z and Y are not connected.

Lemma 5.4. *Let $f : Y \to X$ and $h : Z \to X$ be finite coverings over a connected space X and let $g : Z \to Y$ be an X-morphism $f \circ g = h$.*

For $x \in X$ let $g_x : h^{-1}(x) \to f^{-1}(x)$ be the fiber map. If one of them is bijective, then all are and g is a homeomorphism.

Proof. According to Cor. 5.3 g is a locally finite covering. If we had $Y - g(Z) \neq \varnothing$ then this open and closed set would imply $f(Y - g(Z)) = X$ and a $y \in Y - g(Z)$ with $f(y) = x$ would contradict the surjectivity of g_x. Therefore we have $g(Z) = Y$ and for all $y \in Y$ we must have $\#g^{-1}(y) \geq 1$. Now for $x' \in X$ we get $\deg h = \sum_{y \in f^{-1}(x')} \#g^{-1}(y) \geq \#f^{-1}(x') = \deg f = \deg h$, thus $\forall y \in Y\ \#g^{-1}(y) = 1$. □

Lemma 5.5. *Let X be connected, $f : Y \longrightarrow X$ a finite covering.*

Then we have $Y = Z_1 \dot\cup \cdots \dot\cup Z_r$ where Z_i are the non–empty connected components of Y, and $f_i : Z_i \longrightarrow X$, $f_i = f|Z_i$, are surjective finite coverings.

Proof. Without restriction assume $Y \neq \varnothing$ (otherwise $r = 0$). Consider the open and closed subsets $\varnothing \neq Z \subset Y$. $f(Z) = X$, as X is connected and $Z \longrightarrow X$ is a finite covering. If $Z' \subset Z$ and $Z' \cap f^{-1}(x) = Z \cap f^{-1}(x)$, then $Z' = Z$ by the previous Lemma. There are minimal $Z \neq \varnothing$ and these must be connected. This signifies the finite many minimal Z's are the connected components of Y – and all is done. □

6. Galois coverings

Definition 6.1. A finite covering $f : Y \longrightarrow X$ of connected spaces is called *Galois*[1] with group $G = G(Y/X) := \mathrm{Aut}(Y/X)$, if one of the following equivalent conditions is satisfied:

$$\exists\, y \in Y \quad e_y : G \longrightarrow f^{-1}(f(y)) \text{ is bijective} \qquad (1)$$
$$\sigma \longmapsto \sigma(y)$$

$$\forall\, y \in Y \quad e_y : G \longrightarrow f^{-1}(f(y)) \text{ is bijective} \qquad (2)$$

$$e : G \times Y \xrightarrow{\sim} Y \underset{X}{\times} Y \qquad (3)$$
$$(\sigma, y) \longmapsto (\sigma(y), y)$$

Proof. (of equivalence) (3) ⇒ (2) ⇒ (1) is evident. If (1) holds, apply Lemma 5.4 to the diagram (3) $/Y$. □

Theorem 6.1. *Let $f : Y \longrightarrow X$ be a finite covering of connected spaces $\neq \varnothing$. Then there exists a finite Galois covering $h : Z \longrightarrow X$ such that*

$$e : Z \times \mathrm{Hom}_X(Z, Y) \xrightarrow{\sim} Z \underset{X}{\times} Y \qquad /Z$$
$$(z, g) \longmapsto (z, g(z))$$

and any $T \to X$ with this property, i.e.

$$T \times \mathrm{Hom}_X(T, Y) \xrightarrow{\sim} T \underset{X}{\times} Y$$

factors thru Z: $T \longrightarrow Z \xrightarrow{h} X$.

Proof. Let $x \in X$ and $f^{-1}(x) = \{y_1, \ldots, y_n\}$, $n = \deg f$. Choose

$$Z \subset (Y/X)^n := Y \underset{X}{\times} \cdots \underset{X}{\times} Y \xrightarrow{p_i} Y$$

to be the connected component of $(y_1, \ldots, y_n)$ and $h : Z \longrightarrow X$ canonical.

By Lemma 3.3 e is injective, but the fiber over $(y_1, \ldots, y_n) \in Z$ is mapped surjectively onto $f^{-1}(x)$: $\mathrm{Hom}_X(Z, Y) \xrightarrow{\sim} f^{-1}(x)$, as $p_i \in \mathrm{Hom}_X(Z, Y)$, hence e is bijective by Lemma 5.4.

It remains to be shown that Z/X is Galois. Let $z \in h^{-1}(x)$, we have $p_i(z) \in f^{-1}(x)$, so $p_i(z) = y_{\sigma(i)}$ for some permutation $\sigma \in \mathbf{S}_n$. Interpret σ as a morphism $\sigma : (Y/X)^n \longrightarrow (Y/X)^n$. Since $Z \cap \sigma(Z) \neq \varnothing$ we must have $Z = \sigma(Z)$, and thus $\sigma \in G(Z/X)$ with $\sigma(y_1, \ldots, y_n) = z$ and

$$e_{(y_1, \ldots, y_n)} : G(Z/X) \xrightarrow{\sim} h^{-1}(x)$$

[1] also *normal*

is bijective.

The assertion for T follows at once, since

$$\begin{aligned} T &\longrightarrow (Y/X)^n \\ t &\longmapsto (\alpha_1(t), \dots, \alpha_n(t)) \end{aligned}$$

has image Z, if $\mathrm{Hom}_X(T, Y) = \{\alpha_1, \dots, \alpha_n\}$ has been suitably numbered. □

Lemma 6.2. *Let $f : Y \longrightarrow X$ be Galois, then $G(Y/X)$ operates simply transitive on $\mathrm{Hom}_X(Z, Y)$ for any $h : Z \longrightarrow X$.*

Proof. Without restriction assume $\mathrm{Hom}_X(Z, Y) \neq \varnothing$, let $g : Z \longrightarrow Y$ be such that $f \circ g = h$. Let $z \in Z$, $y = g(z)$, $x = f(y) = h(z)$ and consider

$$\begin{array}{ccccc} G(Y/X) & \hookrightarrow & \mathrm{Hom}_X(Z, Y) & \hookrightarrow & f^{-1}(x) \\ \rho & \longmapsto & \rho \circ g & \longmapsto & \rho(y) \end{array}$$

The injectivity of these mappings follows from Lemma 3.3, the surjectivity of the composed mapping implies $G(Y/X) \xrightarrow{\sim} \mathrm{Hom}_X(Z, Y)$. □

Theorem 6.3. *In the situation $h : Z \xrightarrow{g} Y \xrightarrow{f} X$ let Z/X be Galois. Then Z/Y is Galois, and Y/X is Galois exactly if $G(Z/Y) \lhd G(Z/X)$ is a normal subgroup. Moreover $G(Z/X)$ operates transitively on $\mathrm{Hom}_X(Z, Y)$. In the Galois case we have canonically*

$$G(Y/X) \xrightarrow{\sim} \mathrm{Hom}_X(Z, Y) \xrightarrow{\sim} G(Z/X)/G(Z/Y)$$

Proof. Let $z \in Z$, $y = g(z)$, $x = f(y)$ and consider the diagram

$$\begin{array}{ccc} G(Z/X) & \xrightarrow{\sim} & h^{-1}(x) \\ \uparrow & & \uparrow \\ G(Z/Y) & \hookrightarrow & g^{-1}(y) \end{array} \qquad \tau \mapsto \tau(z)$$

If $\tau(z) \in g^{-1}(y)$, then we have $g \circ \tau(z) = g(z)$, hence by Lemma 3.3 $g \circ \tau = g$, i.e. $\tau \in G(Z/Y)$ and therefore Z/Y is Galois. Furthermore the isotropy group of $g \in \mathrm{Hom}_X(Z, Y)$ in $G(Z/X)$ is exactly $G(Z/Y)$, so

$$G(Z/Y) \backslash G(Z/X) \hookrightarrow \mathrm{Hom}_X(Z, Y)$$

Now, the set on the right has at most $\deg f = \deg Y/X$ elements (Lemma 3.3) and the set on the left has exactly $\deg h / \deg g = \deg f$ elements, which implies $\mathrm{Hom}_X(Z, Y) = \{g \circ \tau \mid \tau \in G(Z/X)\}$.

Now let us investigate the case Y/X Galois: then by Lemma 6.2 $G(Y/X) \xrightarrow{\sim} \mathrm{Hom}_X(Z, Y)$, $\rho \mapsto \rho \circ g$ is bijective. This gives us a canonical mapping

$$\begin{array}{ccc} G(Z/X) & \longrightarrow & G(Y/X) \\ \tau & \longmapsto & \rho \end{array} \qquad \text{where } \rho \circ g = g \circ \tau$$

and we see immediately that this is a homomorphism. The kernel $G(Z/Y)$ is therefore a normal subgroup.

Now let $G(Y/X)$ be a normal subgroup and let us show that Y/X is Galois, that is $\#G(Y/X) = \deg f$. Let $\tau \in G(Z/X)$, $y \in Y$ be given. For any two $z, z' \in g^{-1}(y)$ there is $\sigma \in G(Z/Y)$ with $z' = \sigma(z)$. By assumption $\tau \sigma \tau^{-1} \in G(Z/Y)$, hence $g \circ \tau \sigma = g \circ \tau$, and $g(\tau(z')) = g(\tau(z))$ and $g \circ \tau$ is constant on the fiber, so that the definition $\rho(y) := g(\tau(z))$, for any $z \in g^{-1}(y)$ is meaningful. This shows the surjectivity of $G(Y/X) \longrightarrow \mathrm{Hom}_X(Z, Y)$. □

Now let a connected space $Y \neq \varnothing$ be given with a finite group $G < \mathrm{Aut}(Y)$ of homeomorphisms. Let $X := G\backslash Y$ be the orbit space, the quotient mapping $f : Y \longrightarrow X$ is open and proper. Furthermore f is separated exactly if

$$\forall\, y \in Y\ \exists\, V \in \mathfrak{V}(y) \text{ such that } \forall \sigma \in G - G_y \quad V \cap \sigma(V) = \varnothing$$

Under this condition G is said to operate on Y *discontinuously*.

For f to be étale it is necessary and sufficient that the operation be fixpoint free. We conclude:

Theorem 6.4. *Let $G \subset \mathrm{Aut}(Y)$ be a finite group, which operates discontinuously and without fixpoints on a connected space $Y \neq \varnothing$, let $X := G\backslash Y$. Then Y is a Galois covering of X with Galois group $G(Y/X) = G$.*

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