TOPOLOGY, SHEAVES AND FLAT DESCENT

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Abstract. This paper contains some talks given in the Seminar on Algebraic Geometry in Summer 1978 at the Faculty of Mathematics in Bielefeld. Its main purpose is a fast introduction to flat descent theory. The audience is expected to have some familiarity with schemes. Full references to the literature are given where necessary.

1. Introduction

1.1. **Galois–descent.** Let's consider vector spaces over Galois field extensions. Let E/F be a Galois extension with group G:

$$\begin{array}{ccc} E & V & E-\text{vector space} \\ \Big|\, G & & \\ F & W & F-\text{vector space} \end{array}$$

Proposition 1.1. (1) *Let W be given; then $V := W \otimes_F E$ has a (continuous) G–operation*

$$\begin{array}{ccc} G\times V & \longrightarrow & V \\ \sigma(w\otimes x) & = & w\otimes\sigma(x) \end{array} \qquad w\in W, x\in E, \sigma\in G$$

and we have

$$W = V^G$$

(2) *Let V be given with (continuous) G–operation*

$$\begin{array}{ccc} G\times V & \longrightarrow & V \\ \sigma(x.v) & = & \sigma(x).\sigma(v) \end{array} \qquad \text{such that for } x\in E,\, v\in V$$

Put $W = V^G$, then we have

$$W\otimes_F E \xrightarrow{\sim} V$$

Proof. As to 1: $W \subset V^G$ is obvious and the other direction can best be seen with a normal basis of E/F, i.e. $E = \sum_{\sigma\in G} F.\sigma(x)$ for a suitable $x \in E$.

As to 2: Surjectivity: let $v \in V$, without restriction let E/F be finite (replace E by E^H, where H is an open normal subgroup of G, which leaves v invariant).

Let $E = Fe_1 + \cdots + Fe_n$, $n = [E : F]$, $G = \{\sigma_1, \ldots, \sigma_n\}$; and define $w_i := \sum_{j=1}^{n} \sigma_j(e_i.v) \in W$ by construction. The matrix $\big(\sigma_j(e_i)\big)_{1\le i,j\le n} \in M_n(E)$ is invertable (by the Dedekind lemma on linear independance of characters), let (b_{ik})

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be its inverse, then

$$\sum_{i=1}^{n} b_{ik} w_i = \sum_{i,j} b_{ik} \sigma_j(e_i v) = \sigma_k(v)$$

and for one k we have $\sigma_k = \text{id}$, so the above sum yields $= v$ qed. Injectivity follows like this: Let U be the kernel

$$0 \to U \longrightarrow W \otimes_F E \longrightarrow V \to 0$$

Since $U^G = 0$ by part 1 and $U^G \otimes_F E \longrightarrow U \to 0$ we must have $U = 0$! □

1.2. **Möbius strip.** From the homotopy classification of real vector bundles on the circle group $\mathbf{S}_1$ we know that there are up to isomorphism exactly <u>two</u> line bundles on $\mathbf{S}_1$:

$$\begin{array}{ccccccccc} & & 1 & \to & \mathbb{R}^\times & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathbf{S}_1 & \to & 1 \\ & & & & & & t & \longmapsto & \frac{t^2}{|t|^2} & & \end{array}$$

$$\begin{array}{ccccccccccc} \phi: & \mathbf{S}_1 & \longrightarrow & \mathbf{P}_1(\mathbb{R}) & \simeq & \mathbb{R}^2 - 0/\mathbb{R}^\times & = & \mathbb{C}^\times/\mathbb{R}^\times & = & \mathbf{S}_1 \\ & t & \longmapsto & \mathbb{R}.t & \simeq & t.\mathbb{R}^\times & & \longmapsto & & t^2 \end{array}$$

The non–trivial bundle is a well–known one

$$\begin{array}{ccc} H & = & \{(L,v) \in \mathbf{P}_1(\mathbb{R}) \times \mathbb{R}^2 | v \in L\} \\ \downarrow & & \text{canonical (Hopf}-\text{) bundle} \\ \mathbf{P}_1(\mathbb{R}) & & \end{array}$$

Now consider the 'ball'–bundle over $\mathbf{S}_1$: $B = \{(L,v) \in H \mid |v| \leq 1\} \subset H$. The induced bundle (fibre product) over $\mathbf{S}_1$

$$\phi^* B \simeq \{(t,v) \in \mathbf{S}_1 \times \mathbb{R}^2 \mid v \in \mathbb{R}t, |v| \leq 1\}$$

is trivial with trivialization

$$\begin{array}{ccc} \mathbf{S}_1 \times [-1,+1] & \xrightarrow{\sim} & \phi^* B \\ (t,u) & \longmapsto & (t, t.u) \end{array}$$

and the following (commutative) diagram

$$\begin{array}{ccc} \mathbf{S}_1 \times [-1,+1] & \longrightarrow & \mathbf{S}_1 \times [-1,+1]/\pm 1 \\ \downarrow & & \downarrow \\ \mathbf{S}_1 & \longrightarrow & \mathbf{S}_1/\pm 1 \end{array}$$

1.3. **Descent.** The theory of descent studies situations that generalize both examples above. Consider the following setup (in some category $\mathcal{C}$, say)

$$\begin{array}{ccc} X' & & \\ \downarrow & & \\ S' & \xrightarrow{f} & S \end{array}$$

The question asked is: under which conditions on f and X'/S' exists an X/S such that

$$X' \simeq S' \underset{S}{\times} X = f^* X$$

or in words: when can you descend from X' to a real X ?

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

The connection with the Möbius strip above is clear.
For Galois, let $S' = \operatorname{Spec} E \longrightarrow S = \operatorname{Spec} F$, $X' = \operatorname{Spec} \mathbf{S}_E(V)$, $X = \operatorname{Spec} \mathbf{S}_F(W)$ (vector bundles) and the symmetric algebra is $\mathbf{S}_E(V) = \mathbf{S}_F(W) \otimes_F E$.

One also considers more general situations: any some objects $\mathcal{F}'$ "on" S' and formalizes this in the notion of a "fibered category" $\mathcal{E} \longrightarrow \mathcal{C}$, $\mathcal{E}_S \longrightarrow S$ for $S \in \operatorname{Ob} \mathcal{C}$.

Additional motivations are to be found in the literature: read on in SGA 1 [7], chap VI and introduction of VIII by Grothendieck, in SGA 3 [10], chap IV, no 2 "morphisms of descent" by Demazure and in SGA $4\frac{1}{2}$ [3], chap I "topologies de Grothendieck" by Deligne.

1.4. **Historical notes.** In 1956 A. Weil formulates and proves for the first time the principle of Galois descent in his paper [13]. This principle is also exposed in 1959 in the book by Serre [12, V § 4, 20].

In the same year A. Grothendieck formulates the general theory of descent [6]. He takes special care of purely inseparable field extensions that can't be dealt with by Galois methods alone.

In the sequel emerged several special 'localisation techniques', which in 1962 led to the definition of Grothendieck's topologies ([1]).

In May 1963 the Bourbaki–seminar article "Analysis Situs" by J. Giraud was published [4], which gave the elegant definition of sieves and in which all familiar spectral sequences of (Grothendieck–) topology are derived from the same theorem of Homological Algebra (which is $R^*(f \circ g) \Leftarrow R^*f \circ R^*g$, if g maps injectives to f–acyclic objects).

Nowadays it appears more natural to formulate the theory of descent in the topological language. This becomes particularly transparent in Deligne's Čech point of view in SGA $4\frac{1}{2}$ [3, I 3.2] : the glue condition of descent theory is exactly the one of sheaf theory, cf. Godement [5].

Additional Literature: Michael Artin's notes on a seminar [1] is still a very readable introduction, especially for étale topology. The undaunted will of course consult the full story (> 1500 pages !) in SGA 4 [11].

2. Topology and Sheaves

Definition 2.1. A *presheaf* (of sets) on a category $\mathcal{C}$ is nothing but a normal contravariant functor $P : \mathcal{C}^\circ \longrightarrow \mathcal{S}ets$.

The category of all presheaves is denoted by $\widehat{\mathcal{C}}$. By the Yoneda–Lemma the category $\mathcal{C}$ can be identified with a full subcategory $\mathcal{C} \subset \widehat{\mathcal{C}}$.

Definition 2.2. A *sieve* of an object $X \in \operatorname{Ob} \mathcal{C}$ is just a subfunctor $R \subset X$ in the category $\widehat{\mathcal{C}}$.

If $R_1 \subset R_2$, then we say R_1 is *finer* than R_2. For a sieve R of X and a morphism $f : Y \to X$ the pull–back $f^*R = R \underset{X}{\times} Y \subset Y$ is a sieve of Y.

Note. X itself is a sieve (obviously the coarsest possible).

2.1. **Topology.**

Definition 2.3. A *topology* on the category $\mathcal{C}$ consists of a collection for each $X \in \operatorname{Ob}\mathcal{C}$ of a set $J(X)$ of sieves of X such that the following axioms hold:

(T0) $X \in J(X)$

(T1) (stability under base change) $R \in J(X),\ f : Y \to X \Longrightarrow f^*R \in J(Y)$

(T2) (locality) Let R, S be sieves of X and $R \in J(X)$. If for each Y, and all $y \in R(Y)$ we have $y^*S \in J(Y)$, then also $S \in J(X)$.

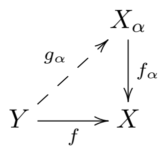
Definition 2.4. A Category together with a topology $(\mathcal{C}, J)$ is called a *situs*.

Exercise 2.1. $R_1, R_2 \in J(X) \Longrightarrow R_1 \cap R_2 \in J(X)$.

A sieve $R \in J(X)$ is called a *refinement* of X or *covering* X (for the topology J).

A family $\mathfrak{U} = (X_\alpha \xrightarrow{f_\alpha} X)_\alpha$ is called *covering* (for J), if the generated sieve $R_{\mathfrak{U}}$ is covering: $R_{\mathfrak{U}} \in J(X)$. Here we set

$$R_{\mathfrak{U}}(Y) := \{f : Y \to X \mid \exists\, g_\alpha : Y \to X_\alpha \text{ with } f = f_\alpha \circ g_\alpha\}$$



A topology is *finer* ($\succ$) than another, the more covering sieves there are:

$$J_1 \succ J_2 \Longleftrightarrow \forall X\ J_1(X) \supset J_2(X)$$

There is for any family J_α of topologies an upper and a lower bound:

- $\sup J_\alpha =$ coarsest topology, that is finer than each J_α
- $\inf J_\alpha =$ finest topology, that is coarser than each J_α

Obviously we have $\inf J_\alpha(X) = \bigcap_\alpha J_\alpha(X)$, as the intersection satisfies already all the axioms of a topology.

If you are given for any $X \in \operatorname{Ob}\mathcal{C}$ families of morphisms $(X_\alpha \to X)$, there are topologies for which these families are covering, for example the coarsest of all these. The covering sieves of this topology are difficult to determine. With the following definition this determination becomes simpler.

Definition 2.5. A *pre–topology* on a category $\mathcal{C}$ is given by sets $Cov(X)$ of families of morphisms with target X with the following axioms:
(PT0) $id_X \in Cov(X)$
(PT1) For $\mathfrak{U} = (X_\alpha \to X)_\alpha \in Cov(X)$ and any $Y \xrightarrow{f} X$ exists $X_\alpha \underset{X}{\times} Y$ and $f^*\mathfrak{U} = (X_\alpha \underset{X}{\times} Y \to Y)_\alpha \in Cov(Y)$
(PT2) If $(X_\alpha \to X)_\alpha \in Cov(X)$ and for each α $(X_{\alpha\beta} \to X_\alpha)_\beta \in Cov(X_\alpha)$, then we have $(X_{\alpha\beta} \to X)_{\alpha\beta} \in Cov(X)$.

Exercise 2.2. For the topology J generated by a pre–topology we have

$$R \in J(X) \Longleftrightarrow \exists\, \mathfrak{U} \in Cov(X) \text{ with } R_{\mathfrak{U}} \subset R$$

Hint: “$\Leftarrow$” is clear. Show that the R, which have a finer $R_{\mathfrak{U}}$, build already a topology — this is the coarsest such.

2.2. **Sheaves.** Let $\mathcal{C}$ be a situs. A presheaf $\mathcal{F}$ is called a *sheaf* (with respect to the topology J), if for all $X \in \operatorname{Ob}\mathcal{C}$, $R \in J(X)$

$$\begin{array}{ccc} \operatorname{Hom}_{\widehat{\mathcal{C}}}(X, \mathcal{F}) & \xrightarrow{\sim} & \operatorname{Hom}_{\widehat{\mathcal{C}}}(R, \mathcal{F}) \\ s & \longmapsto & s|R \end{array}$$

By abuse of notation it is generally written $P(Q) := \operatorname{Hom}_{\widehat{\mathcal{C}}}(Q, P)$, which by Yoneda is consistent in case $Q \in \operatorname{Ob}\mathcal{C}$.

The sheaf–condition then simply states $\mathcal{F}(X) \simeq \mathcal{F}(R)$ for all R covering X.

The sheaves build a full sub–category $\widetilde{\mathcal{C}}$ of the presheaves $\widehat{\mathcal{C}}$.

A category of sheaves is sometimes also called a *topos.* Comparing topologies: when $J_1 \succ J_2$, then for the topos it follows that $\widetilde{\mathcal{C}}_1 \subset \widetilde{\mathcal{C}}_2$.

Theorem 2.1. *When the topology is defined by a pre–topology, then $\mathcal{F}$ is a sheaf if and only if $\forall X, \forall \mathfrak{U} = (X_\alpha \to X)_\alpha \in Cov(X)$ the following diagram is exact:*

$$\mathcal{F}(X) \to \prod_{\alpha} \mathcal{F}(X_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(X_\alpha \underset{X}{\times} X_\beta)$$

This theorem relates the above definition to the familiar one. We will first need a lemma and postpone the proof.

Lemma 2.2. *For $\mathfrak{U} = (X_\alpha \xrightarrow{f_\alpha} X)_\alpha \in Cov(X)$ we have canonically (i.e. functorially in $\mathcal{F} \in \widehat{\mathcal{C}}$)*

$$\begin{array}{ccc} \operatorname{Hom}_{\widehat{\mathcal{C}}}(R_{\mathfrak{U}}, \mathcal{F}) & \simeq & \operatorname{Ker}\Big(\prod_{\alpha} \mathcal{F}(X_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(X_\alpha \underset{X}{\times} X_\beta)\Big) \\ s & \longmapsto & (s_\alpha)_\alpha \end{array}$$

Proof. As $f_\alpha : X_\alpha \to X$ is in $R_{\mathfrak{U}}(X_\alpha)$, we can define $s_\alpha := s(X_\alpha) f_\alpha \in \mathcal{F}(X_\alpha)$. From the following commutative diagram

$$\begin{array}{ccc} X_\alpha \underset{X}{\times} X_\beta & \longrightarrow & X_\beta \\ \big\downarrow & & \big\downarrow f_\beta \\ X_\alpha & \xrightarrow{f_\alpha} & X \end{array}$$

we read, that $s_\alpha | X_\alpha \underset{X}{\times} X_\beta = s_\beta | X_\alpha \underset{X}{\times} X_\beta = s(X_\alpha \underset{X}{\times} X_\beta) f_{\alpha\beta}$, where $f_{\alpha\beta}$ is the diagnal map $f_{\alpha\beta} : X_\alpha \underset{X}{\times} X_\beta \longrightarrow X$.
This defines the map $\mathcal{F}(R_{\mathfrak{U}}) \to \operatorname{Ker}$.

Let now $(s_\alpha)_\alpha \in \operatorname{Ker}$ be given. Pick a $\varphi \in R_{\mathfrak{U}}(Y)$ and let $\varphi = f_\alpha \circ \varphi_\alpha$ be a factorization

$$\begin{array}{ccc} Y & \xrightarrow{\varphi_\alpha} & X_\alpha \\ & \searrow^{\varphi} & \big\downarrow f_\alpha \\ & & X \end{array}$$

Then we define

$$s(Y)\varphi := \mathcal{F}(\varphi_\alpha) s_\alpha$$

which is well–defined, as the (s_α) satisfy the glueing condition $s_\alpha | X_\alpha \underset{X}{\times} X_\beta = s_\beta | X_\alpha \underset{X}{\times} X_\beta$.

This construction is functorial (and is similar to the one in the Yoneda–Lemma). □

Proof of Theorem 2.1. One direction in the assertion of the theorem is clear: When $\mathcal{F}$ is a sheaf, then in particular for the sieves of the form $R_{\mathfrak{U}}$, $\mathfrak{U} \in Cov(X)$, we have

$$\mathcal{F}(X) \simeq \mathcal{F}(R_{\mathfrak{U}})$$

and the right hand side is the kernel, according to the Lemma 2.2.

Conversely, let the diagram be exact, that is for all $\mathfrak{U} \in Cov(X)$

$$\mathcal{F}(X) \simeq \mathcal{F}(R_{\mathfrak{U}})$$

and we have to show, that for all coverings R of X the canonical map

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(R)$$

is bijective.

There $\exists\, \mathfrak{U}$ such that $R \supset R_{\mathfrak{U}}$, by definition of the topology, and

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(R) \longrightarrow \mathcal{F}(R_{\mathfrak{U}})$$

is bijective, hence $\mathcal{F}(X) \hookrightarrow \mathcal{F}(R)$ is injective. It remains to show surjectivity.

Let a local section $s_R \in \mathcal{F}(R)$ be given, its restriction to the finer covering sieve $R_{\mathfrak{U}}$ comes from a uniquely determined global section $s \in \mathcal{F}(X)$, that is

$$\begin{gathered} \forall\, T \xrightarrow{\psi} X \text{ with } \psi \in R_{\mathfrak{U}}(T) \text{ we have} \\ s|T = \mathcal{F}(\psi)s = s_R(T)\psi \in \mathcal{F}(T) \end{gathered} \tag{1}$$

and it remains to be shown, that this implies $s|R = s_R$, the global section coincides with the locally given one on the covering R.

The idea is, that it suffices to prove this locally !

Take Y, $\varphi \in R(Y)$ at random and consider the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ {\scriptstyle g_\alpha}\uparrow & & \uparrow{\scriptstyle f_\alpha} \\ Y_\alpha & \longrightarrow & X_\alpha \end{array}$$

With $\mathfrak{U} = (X_\alpha \to X)_\alpha \in Cov(X)$ we have $\mathfrak{V} = \varphi^*\mathfrak{U} = (Y_\alpha \to Y)_\alpha \in Cov(Y)$ and $\mathcal{F}(Y) \xrightarrow{\sim} \mathcal{F}(R_{\mathfrak{V}})$ by assumption.

For $s_R(Y)\varphi = s|Y$ to hold it suffices to show

$$\forall\, \alpha \quad s_R(Y_\alpha)(\varphi \circ g_\alpha) = s|Y_\alpha \tag{2}$$

Now, by construction, $\varphi \circ g_\alpha \in R_{\mathfrak{U}}(Y_\alpha)$, and therefore this relation (2) follows from the equation (1) above. □

The theorem has been proven here in all detail, as its proof illustrates the intuitive–geometric argumentation that is the mark of the language of sheaves (everything is exactly as expected from topology, see [5]).

2.3. **Examples.**

Example 2.1 (Trivial Topologies)**.** Trivial examples for topologies are the *discrete* topology

$$J_{discr}(X) = \{\text{all sieves of } X\}$$

and the *chaotic* topology

$$J_{chaos}(X) = \{X\}$$

It is obvious that the chaotic sheaves make up all of the co–functors: $\widetilde{\mathcal{C}}_{chaos} = \widehat{\mathcal{C}}$.

Example 2.2 (Topological Spaces)**.** Let X be a topological space, $\mathcal{C} = X_{top} = Open(X)$ the ordered set of its open subsets, interpreted in the usual way as a category.

A natural pre–topology is defined by the open coverings of the topological space:

$$\mathfrak{U} = (U_\alpha \hookrightarrow U)_\alpha \in Cov(U) \Longleftrightarrow \bigcup_\alpha U_\alpha = U$$

It is easy to see, that $R_{\mathfrak{V}} \subset R_{\mathfrak{U}} \Longleftrightarrow \mathfrak{V}$ is finer than $\mathfrak{U}$. The sieves are the natural generalizations of equivalence classes of coverings in the ordinary sense.

The topos $\widetilde{\mathcal{C}} = \widetilde{X}_{top}$ is the familiar category of sheaves on X.

A continuous map $f : Y \longrightarrow X$, i.e. a $f^{-1} : X_{top} \longrightarrow Y_{top}$, induces a map on presheaves respecting the topology of the topological situs $f_\bullet : \widehat{Y}_{top} \longrightarrow \widehat{X}_{top}$, so this induces a map $f_* : \widetilde{Y}_{top} \longrightarrow \widetilde{X}_{top}$. It is an interesting exercise to characterize the continuous maps $f : Y \longrightarrow X$ that induce isomorphisms on the topological topos $f_* : \widetilde{Y}_{top} \xrightarrow{\sim} \widetilde{X}_{top}$. These morphisms are not necessarily homeomorphisms, but almost so: they are called *quasi–homeomorphisms* (see SGA [11, IV, Cor. 4.2.4], or EGA I [8, 0.2.7, 0.3.8]).

Example 2.3 (Sets)**.** $\mathcal{C} = \mathcal{S}ets$ and $(X_\alpha \xrightarrow{f_\alpha} X)$ is covering, when $\bigcup f_\alpha(X_\alpha) = X$.

Here the sheaves are no surprise: $\mathcal{F}$ is a sheaf $\Leftrightarrow$ $\mathcal{F}$ is representable (by $\mathcal{F}$(point)).

Example 2.4 (G–sets)**.** Let G be a group, $\mathcal{C} = (G\text{–}\mathcal{S}ets)$, with the same covering refinements as above. Again we have:

$\mathcal{F}$ is a sheaf $\Leftrightarrow$ $\mathcal{F}$ is representable

Proof. Show

$$(*) \qquad \begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\sim} & \mathrm{Hom}_G(X, \mathcal{F}(G)) \\ s & \longmapsto & \tilde{s} \end{array}$$

defined by $\tilde{s}(x) = \mathcal{F}(f_x)s$, where $f_x : G \to X$ is the orbit mapping $f_x(g) = g.x$. Here we make $\mathcal{F}(G)$ into a (left) G–set by transport of structure: when for $g \in G$ we denote right translation on G by $\rho(g) : G \longrightarrow G$, then $\mathcal{F}(\rho(g)) : \mathcal{F}(G) \longrightarrow \mathcal{F}(G)$ defines the G–law.

To prove the representability ($*$) of a sheaf, apply the sheaf condition to the following covering family: $(f_\alpha : X_\alpha \to X)_{\alpha \in G\backslash X}$; $X_\alpha = G$ and $f_\alpha : X_\alpha \longrightarrow X$ is the orbit map. □

Of course 2.3 is the special case $G = \{e\}$.

Example 2.5 (Flat Topology)**.** For $\mathcal{C}$ = (Schemes), the *flat* topology (*fpqc*) is defined by the pre–topology of the following families

$$\begin{gathered}(f_\alpha : X_\alpha \longrightarrow X)_\alpha \quad \text{with} \\ \forall\, \alpha \quad f_\alpha \text{ is flat and quasi–compact} \\ \bigcup f_\alpha(X_\alpha) = X\end{gathered}$$

For a more detailed characterization, see section 3.3.

Example 2.6 (Étale Topology)**.** For $\mathcal{C}$ = (Schemes), the *étale* topology (*ét*) is analogously defined, only restricting the f_α to be étale.

Example 2.7 (Zariski Topology)**.** For $\mathcal{C}$ = (Schemes), the *Zariski* topology (*Zar*) is similarly defined, now requiring all f_α to be open immersions.

As $J_{fpqc} \succ J_{ét} \succ J_{Zar}$ we have for the corresponding topoi of schemes

$$\widetilde{Sch}_{fpqc} \subset \widetilde{Sch}_{ét} \subset \widetilde{Sch}_{Zar}$$

A Zariski–sheaf on (Schemes) is also called a 'local functor'.

Example 2.8 (Canonical Topology)**.** Let $\mathcal{F}$ be any presheaf on $\mathcal{C}$ and define

$$J_{\mathcal{F}}(X) := \{R \subset X \mid \forall\, Y \xrightarrow{f} X : \mathcal{F}(Y) \xrightarrow{\sim} \mathcal{F}(f^*R)\}$$

Claim. $J_{\mathcal{F}}$ is a topology and is obviously the finest one, making $\mathcal{F}$ a sheaf.

Proof. This is a good exercise. First show that $R, S \in J_{\mathcal{F}}(X)$ implies $R \cap S \in J_{\mathcal{F}}(X)$ and then prove (T2). (The axioms (T0) and (T1) hold trivially by definition). This is somewhat technical and can be looked up in SGA 4 [11], exposé II 'Topologies et Faisceaux' by J.-L. Verdier, Prop. 2.2. □

Definition 2.6. The *canonical* topology on a category $\mathcal{C}$

$$J_{can}(X) = \bigcap_{Y \in \mathrm{Ob}\,\mathcal{C}} J_Y(X)$$

is the finest topology, in which all representable presheaves are sheaves, therefore

$$\mathcal{C} \subset \widetilde{\mathcal{C}}_{can} \subset \widehat{\mathcal{C}}$$

It may happen, though, that there are *non–representable canonical sheaves*, and $\mathcal{C} \neq \widetilde{\mathcal{C}}_{can}$ (in fact, this is usually the case).

In the examples 2.2 – 2.7 all the topologies have been coarser than the canonical topology, so that in all cases the representable objects have been sheaves, and the sheaf condition is a necessary condition for representability. In particular this statement for the flat topos 2.5

$$(Sch) \subset \widetilde{Sch}_{fpqc}$$

is the main theorem of flat descent (proof in 3.4).

2.4. **Associated sheaf to a presheaf.** To each presheaf P is associated a sheaf in a canonical manner: one glues together locally given sections to 'ideal' global sections, i.e. let us define

$$P^+(X) := \varinjlim_{R \in J(X)} P(R)$$

We will see that repeating twice this procedure suffices to get sheaves !

We recall that $P(R) = \mathrm{Hom}_{\widehat{\mathcal{C}}}(R, P)$ are the natural transformations of functors, by definition.

Let

$$z_R : \mathrm{Hom}(R, P) \longrightarrow P^+(X)$$

be the canonical mapping to the inductive limit, which is interpreted as "glueing sections together on the sieve R".

P^+ is a functor, because of (T1), that is for $Y \to X$ we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(R, P) & \xrightarrow{z_R} & P^+(X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(Y \underset{X}{\times} R, P) & \xrightarrow[z_{Y \underset{X}{\times} R}]{} & P^+(Y) \end{array}$$

Because of (T0) there is $z_X : \mathrm{Hom}(X, P) \longrightarrow P^+(X)$ and this combined with Yoneda gives a map $\ell(X) : P(X) \longrightarrow P^+(X)$, which is functorial in X:

$$\ell : P \longrightarrow P^+$$

This construction is also functorial in P, of course, and if this dependence is important we denote this morphism by

$$\ell_P : P \longrightarrow P^+$$

Note. The functor $L : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}$, $L(P) = P^+$, is left exact, as filtrant inductive limits commute with finite projective limits. The derived functor $R^*L = \check{\mathcal{H}}^*$ is called *Čech cohomology.*

Lemma 2.3. (1) *For any refinement R of X and any $u : R \to P$ we have the commutative diagram:*

$$\begin{array}{ccc} P & \xrightarrow{\ell} & P^+ \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle z_R(u)} \\ R & \hookrightarrow & X \end{array}$$

(2) *For $\forall\, X$, $\forall\, s, t : X \rightrightarrows P$ with $\ell \circ s = \ell \circ t$ we have for the kernel $R := \mathrm{Ker}(s, t)$ that $R \in J(X)$.*

Proof. 1) Let T, $t \in R(T)$ be arbitrary and consider the previous commutative diagram with $Y = T$, taking into account that $T \simeq T \underset{X}{\times} R$, because $t : T \to X$ factors thru $R \subset X$, since $t \in R(T)$. This gives the diagram

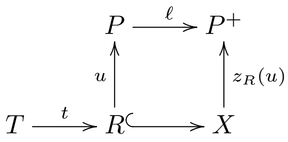
$$\begin{array}{ccc} u \in \mathrm{Hom}(R, P) & \xrightarrow{z_R} & P^+(X) \\ \downarrow & & \downarrow{\scriptstyle P^+(t)} \\ ut \in P(T) & \xrightarrow[\ell(T)]{} & P^+(T) \end{array}$$

and the commutativity translates into

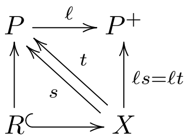
$$P^+(t)z_R(u) = \ell(T)(u.t)$$

$$z_R(u)|_R.t = (\ell \circ u).t$$

As this holds for any t, we get $z_R(u)|_R = \ell \circ u$, qed for 1).



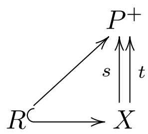
2) This is evident



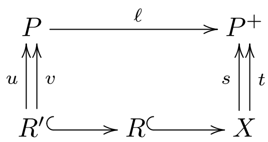
so, by definition of P^+ as a $\varinjlim$, there is a $R' \in J(X)$ on which s, t coincide, thus $R' \subset R$ and $R \in J(X)$. ☐

Proposition 2.4. *$\forall\ X$, $\forall\ R \in J(X)$ we have $P^+(X) \hookrightarrow P^+(R)$ is injective.*

Proof. Let $s, t \in P^+(X)$ with $s|R = t|R$.



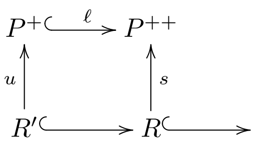
Write $s = z_{R'}(u)$, $t = z_{R'}(v)$, with $R' \subset R$



As $\ell \circ u = \ell \circ v$ there is $R'' \subset R'$ with $u|R'' = v|R''$. Therefore $s = z_{R''}(u|R'') = z_{R''}(v|R'') = t$. ☐

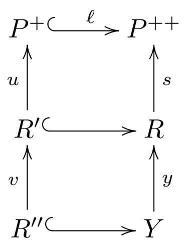
Proposition 2.5. *$\forall\ X$, $\forall\ R \in J(X)$ we have $P^{++}(X) \xrightarrow{\sim} P^{++}(R)$ is bijective, i.e. P^{++} is a sheaf.*

Proof. Surjectivity remains to be shown. Let $s \in P^{++}(R)$ and consider the cartesian product $R' := P^+ \underset{P^{++}}{\times} R$:

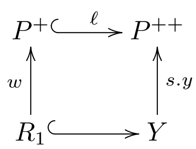


From the previous proposition we know that ℓ is injective here, therefore $R' \subset R \subset X$ is a sieve of X. We will now show that it must be covering: $R' \in J(X)$.

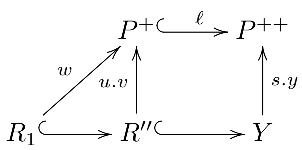
To this end we will apply (T2): take a $Y \xrightarrow{y} R$ and pull back R' to Y:



The global section $s.y \in P^{++}(Y)$ comes from a $w : R_1 \longrightarrow P^+$, $R_1 \in J(Y)$, $s.y = z_{R_1}(w)$ (by definition of $\varinjlim$). By the Lemma 2.3 the diagram

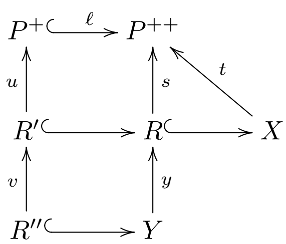


is commutative and R_1 factors thru the fiber product R'',



$R_1 \subset R'' \subset Y$. As $R_1 \in J(Y)$ this implies $R'' \in J(Y)$, and the prerequisites for (T2) are satisfied: hence $R' \in J(X)$.

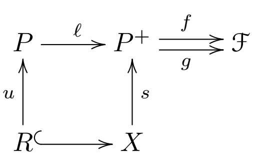
Now $t = z_{R'}(u)$ is our candidate for the global section over X. Look at a big diagram, whose upper right triangle we want to prove to be commutative, and chase a bit around.



Now $t|R' = s|R'$ implies $t|R' \circ v = s|R' \circ v$, i.e. $t|R \circ y|R'' = s \circ y|R''$, which bei Proposition 2.4 implies $t|R \circ y = s \circ y$. As this is true for all y, we have $t|R = s$. □

Proposition 2.6. *For a sheaf* $\mathcal{F}$*:* $\mathrm{Hom}(P^+, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(P, \mathcal{F})$ *under the mapping* $f \mapsto f \circ \ell$.

Proof. For injectivity, let f, $g : P^+ \longrightarrow \mathcal{F}$ be two morphisms with $f \circ \ell = g \circ \ell$. It suffices to show $f(s) = g(s)$ for $\forall\ s \in P^+(X)$, $\forall X$. Now, there exists $\exists\ R \in J(X)$ with $s = z_R(u)$ for a $u : R \to P$



which implies that $fs|R = gs|R \in \mathcal{F}(R)$, which by the sheaf condition implies $fs = gs \in \mathcal{F}(X)$.

For surjectivity, let $h : P \to \mathcal{F}$ be given and consider by functoriality $h^+ : P^+ \to \mathcal{F}^+$ and the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ell_P} & P^+ \\ {\scriptstyle h}\downarrow & \swarrow{\scriptstyle f} & \downarrow{\scriptstyle h^+} \\ \mathcal{F} & \xrightarrow[\ell_{\mathcal{F}}]{} & \mathcal{F}^+ \end{array}$$

Remark that $\ell_{\mathcal{F}} : \mathcal{F} \simeq \mathcal{F}^+$ is an isomorphism for sheaves, so $f = \ell_{\mathcal{F}}^{-1} \circ h^+$ satisfies $f \circ \ell = h$ □

Corollary 2.7. *The functor* $a : \widehat{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}}$

$$P \longmapsto P^{++}$$

is a left adjoint to the inclusion $i : \widetilde{\mathcal{C}} \subset \widehat{\mathcal{C}}$. *Obviously it commutes with finite projective limits (as* $\varinjlim_{R \in J(X)}$ *is filtrant), which makes* a *an exact functor.*

Proof. This is clear from the above. □

Note. Let's consider abelian (pre)sheaves $P : \mathcal{C}^\circ \longrightarrow (Ab)$.

For an object $X \in \operatorname{Ob} \mathcal{C}$, let $v_X : \widehat{\mathcal{C}} \longrightarrow (Ab)$ be the evaluation functor $v_X(P) = P(X)$, which is exact. Then $\Gamma_X = v_X \circ i$ is the usual (left exact) global section functor $\Gamma_X : \widetilde{\mathcal{C}} \longrightarrow (Ab)$, whose derivation is (by definition) the sheaf cohomology $H^*(X, \mathcal{F})$. The functor $P \mapsto H^0(R, P) = \operatorname{Hom}_{\widehat{\mathcal{C}}}(R, P)$ is left exact, whose derivation is the cohomology of the covering $H^*(R, P)$.

Remark that $i \circ a = L \circ L$, $i \simeq L \circ i$. These relations give rise to the classical spectral sequences of cohomology, relating Čech cohomology, cohomology of a covering and sheaf cohomology to one another, see Giraud's "Analysis Situs" [4].

3. Flat Topology

3.1. **Properties of the flat Topology.** Let S be a scheme.

$$S_{flat} = (\widetilde{Sch/S})_{fpqc}$$

is the flat topos over S. Let $X, Y \in (Sch/S)$ be schemes over S. Define the presheaf of sets $\mathcal{H}om_S(Y, X)$ by

$$\begin{aligned} (Sch/S) &\longrightarrow \mathcal{S}ets \\ T/S &\longmapsto \operatorname{Hom}_T(Y \underset{S}{\times} T, X \underset{S}{\times} T) \end{aligned}$$

Theorem 3.1. *The presheaf* $\mathcal{H}om_S(Y, X)$ *is a sheaf for the flat topology.*

This immediately implies: the presheaf of section of X/S

$$\mathbf{\Gamma}(X/S) : T \longrightarrow \Gamma(X \underset{S}{\times} T/T) = \operatorname{Hom}_T(T, X \underset{S}{\times} T)$$

is a flat sheaf. Or similarly, the presheaf $h_{X/S}$ given by

$$(*) \qquad T \longmapsto \operatorname{Hom}_S(T, X)$$

is a flat sheaf ($h_{X/S} \simeq \mathbf{\Gamma}(X/S)$).

Conversely, if (∗) holds, let $(T_\alpha \longrightarrow T)_\alpha$ be flat covering, then so is $Y \underset{S}{\times} T_\alpha \longrightarrow Y \underset{S}{\times} T$, therefore yielding the exact sequence

$$\operatorname{Hom}_S(Y \underset{S}{\times} T, X) \to \prod_\alpha \operatorname{Hom}_S(Y \underset{S}{\times} T_\alpha, X) \rightrightarrows \prod_{\alpha,\beta} \operatorname{Hom}_S(Y \underset{S}{\times} (T_\alpha \underset{T}{\times} T_\beta), X)$$

And as $\mathcal{H}om_S(Y,X)(T) = \mathrm{Hom}_T(Y \underset{S}{\times} T, X \underset{S}{\times} T) \simeq \mathrm{Hom}_S(Y \underset{S}{\times} T, X)$, we get a similar diagram for $\mathcal{H}om_S(Y,X)$, and the assertion for $(*)$ implies Theorem 3.1.

We will prove $(*)$ only for $S = \mathrm{Spec}\,\mathbb{Z}$, from which the relative case will follow. Now, $(*)$ reformulates to

Theorem 3.2. *In the flat topology any scheme is a sheaf.*

The theorem 3.2 is the main theorem of flat descent theory and will be proved with its variants in 3.4, after a more detailed study of the class of morphisms involved has been done in 3.2 and a characterization of the flat topology has been achieved in 3.3.

3.2. **Flat Morphisms.** This section is basically a summary of results and definitions in commutative algebra.

Proposition 3.3. *Let A be a (commutative) ring and E be an A–module. The following conditions are equivalent*

(1) *The tensor product functor is exact:* $? \otimes_A E : (A\text{–}Mod) \longrightarrow (A\text{–}Mod)$
(2) $I \otimes_A E \xrightarrow{\sim} I.E \quad \forall$ *finitely generated ideals* $I \subset A$.
(3) *For any relation* $\sum_{i=1}^n a_i x_i = 0$ *in the module, with* $a_i \in A, x_i \in E$, *there* $\exists\, a_{ij} \in A, y_j \in E$ *such that* $\sum_j a_{ij} y_j = x_i, \sum_{i=1}^n a_i a_{ij} = 0$. *(relations in the module are caused by relations of the coefficients).*

When these conditions are met, the A–module E is called flat.

Proof. See Bourbaki, Algèbre commutative [2, I, § 2,3]. □

3.2.1. *Examples.*

(1) Free modules are flat.
(2) Projective modules are flat.
(3) $A_S = S^{-1}A$ is flat $/A$, where $S \subset A$ is a multiplicative set.
(4) $\forall\, E$ flat $\exists$ inductive system F_α of finitely generated free modules such that $E = \varinjlim_\alpha F_\alpha$ (D. Lazard, see EGA [8, 0, Prop.6.6.24]).

Proposition 3.4. *Let A be a (commutative) ring and E be an A–module. The following conditions are equivalent*

(1) *The tensor product is exact and faithful:* $? \otimes_A E : (A\text{–}Mod) \longrightarrow (A\text{–}Mod)$
(2) E *is flat and* $\forall\, \mathfrak{m}$ *maximal ideals is* $E \neq \mathfrak{m}.E$ *(the support* $\mathrm{supp}\, E = \mathrm{Spec}\, A$*).*

When these conditions are met, the A–module E is called faithfully flat.

3.2.2. *Properties.*

(1) E, F (faithfully) flat $\quad \Rightarrow \quad E \otimes_A F$ (faithfully) flat
(2) E faithfully flat, then we have

$$F \text{ (faithfully) flat} \quad \Longleftrightarrow \quad E \otimes_A F \text{ (faithfully) flat}$$

(3) $A \longrightarrow B$, E an A–module, H a B–module
   (a) E (faithfully) flat $/A \quad \Rightarrow \quad E \otimes_A B$ (faithfully) flat $/B$
   (b) E (faithfully) flat $/A \quad \Leftarrow \quad E \otimes_A B$ (faithfully) flat $/B$
   if B is faithfully flat over A.

   (c) H (faithfully) flat$/B, B$ (faithfully) flat$/A \quad \Rightarrow \quad H$ (faithfully) flat$/A$.
(4) Consider an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

with E'' flat. Then $\forall\, A$–modules M

$$0 \longrightarrow E' \otimes_A M \longrightarrow E \otimes_A M \longrightarrow E'' \otimes_A M \longrightarrow 0$$

is an exact sequence, and

$$E \text{ flat} \Leftrightarrow E' \text{ flat}$$

(5) E is (faithfully) flat over $A \quad \Leftrightarrow \quad \forall\, \mathfrak{p} \in \operatorname{Spec} A\ E_{\mathfrak{p}}$ is (faithfully) flat over $A_{\mathfrak{p}}$ (for $\Leftarrow$ it suffices for $\mathfrak{p}$ maximal).
(6) $\bigoplus_{\mathfrak{m}} A_{\mathfrak{m}}$ is faithfully flat over A

Theorem 3.5. *The following conditions are equivalent*

(1) $\varphi : A \longrightarrow B$ *is faithfully flat*
(2) $A \hookrightarrow B$ *is injective and* B/A *is a flat* A*–module.*
(3) $M \longmapsto M \otimes_A B$ *is exact and* $M \hookrightarrow M \otimes_A B$ *is injective* $\forall\, M \in (A\text{–}Mod)$.
(4) $I \otimes_A B \xrightarrow{\sim} I.B$ *and* $\varphi^{-1}(I.B) = I \quad \forall\, I$ *ideals* $\subset A$.
(5) B *is flat over* A *and* $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ *is surjective.*

Definition 3.1 (Flat morphisms for schemes)**.** $f : Y \longrightarrow X$ is *flat* $:\Leftrightarrow$
$\forall y \in Y \quad \mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,f(y)}$–module.

$f : Y \longrightarrow X$ is *faithfully flat* $:\Leftrightarrow f$ is flat and surjective.

Remark. For X, Y affine the two definition of *flat* coincide:

$$f : Y \longrightarrow X \text{ flat} \Leftrightarrow A = \mathcal{O}_X(X) \xrightarrow{\varphi} \mathcal{O}_Y(Y) = B \text{ flat}$$

This is not completely obvious, as the definition for schemes is local over Y (and not over X).

"$\Rightarrow$": $\forall \mathfrak{q} \subset B$ we have $B_{\mathfrak{q}}$ flat over $A_{\mathfrak{p}}$, where $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, therefore $B_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$, qed.

"$\Leftarrow$": As B is flat over A, for $A_{\mathfrak{p}}$–modules N we have

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} N \cong B_{\mathfrak{q}} \otimes_A N$$

that is $B_{\mathfrak{q}}$ flat over $A_{\mathfrak{p}}$.

3.3. **Characterization of flat Topology.** Remember that topologies can be defined by prescribing families of morphisms that are covering. This will be done now for the flat topology on schemes.

Lemma 3.6. *The coarsest topology for which the following families are covering:*

- *Ia) open coverings* $X = \bigcup U_\alpha$,
- *Ib)* $Y = \operatorname{Spec} B \longrightarrow X = \operatorname{Spec} A$, B *faithfully flat* $/A$.

is identical to the coarsest topology for which the following families are covering:

- *II)* $f_\alpha : X_\alpha \to U_\alpha \subset X$, $\bigcup U_\alpha = X$ *with* f_α *faithfully flat,* X_α *and* U_α *affine,* U_α *open in* X.

This topology is called the flat *topology on schemes and we have*

$$J_{flat}(X) = \{R \subset X \mid \exists\, \mathfrak{U} \text{ of type II } R \supset R_{\mathfrak{U}}\}$$

Proof. Affine open coverings are of type II and any open covering has such a refinement. Therefore $Ia)$ and $Ib)$ are covering for II: $J_I \subset J_{II}$.

Conversely, a family of type II is a composition of those of type I, which implies $J_{II} \subset J_I$, i.e. $J_I = J_{II}$.

Define a set of sieves by $J_0(X) = \{R \subset X \mid \exists \mathfrak{U} \text{ of type } II \; R \supset R_{\mathfrak{U}}\}$, then $J_0 \subset J_{II}$.

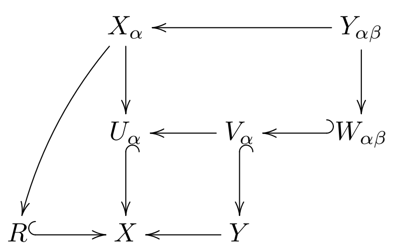
If we can prove J_0 to satisfy the axioms of a topology, then we must have $J_0 = J_{II}$.

We verify the axioms now.

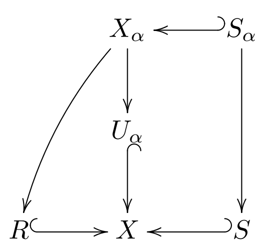
(T0) : is obvious.

(T1) : Let $R \supset R_{\mathfrak{U}}$, $\mathfrak{U} = (X_\alpha \to X)$ of type II, and $f : Y \to X$ be arbitrary.

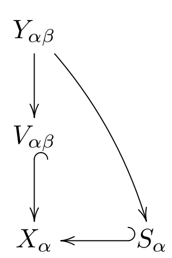
Cover the $V_\alpha = f^{-1}(U_\alpha) \subset Y$ by affine open $W_{\alpha\beta}$, and let $Y_{\alpha\beta} = X_\alpha \underset{U_\alpha}{\times} W_{\alpha\beta}$ then the family $\mathfrak{V} = (Y_{\alpha\beta} \longrightarrow Y)$ is of type II, and $f^*R \supset R_{\mathfrak{V}}$.



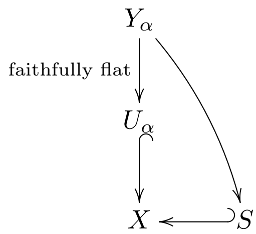
(T2) Let $R \in J_0(X), S \subset X$ be given with the property $\forall f : Y \to R \subset X$ is $f^*S \in J_0(Y)$. We now have to show: $S \in J_0(X)$.



By assumption we have $S_\alpha \in J_0(X_\alpha)$, that is there is a diagram, where without restriction β runs thru a *finite* index set (X_α is quasi–compact):



Let $Y_\alpha = \coprod_\beta Y_{\alpha\beta}$, $V_\alpha = \coprod_\beta V_{\alpha\beta}$. These are affine schemes and moreover $Y_\alpha \to V_\alpha$ is faithfully flat, as well as $V_\alpha \to X_\alpha$, which gives us the diagram



$\Rightarrow S \in J_0(X)$.

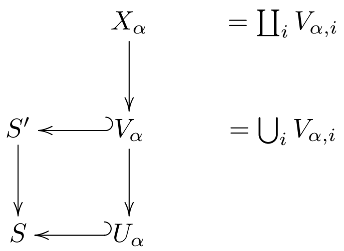
□

Lemma 3.7. *Recall that* $\mathfrak{U} = (X_\alpha \to X)_\alpha$ *is a flat covering, if* $R_{\mathfrak{U}} \in J_{flat}(X)$ *(defined in section 2.1).*

- *III) A surjective family* $f_\alpha : X_\alpha \longrightarrow X$*, with* f_α *flat and open, is a flat covering.*
- *IV) A faithfully flat and quasi–compact morphism* $p : S' \longrightarrow S$ *is a flat covering.*

Remark. The families $III + IV$ generate the flat topology as well.

Proof. For IV: cover $S = \bigcup U_\alpha$, U_α open affine, $\Rightarrow$ $V_\alpha = p^{-1}(U_\alpha)$ is quasi–compact. There is a *finite* open affine covering of the V_α, the disjoint sum X_α of which is affine and faithfully flat over V_α:



and $\mathfrak{U} = (X_\alpha \to X)_\alpha$ is a refinement of type II of S'/S.

A similar reasoning applies to III (see SGA [10, IV, Prop. 6.3.1]). □

3.4. **Characterization of flat Sheaves.** This section contains the "reduction to the affine case".

Theorem 3.8. $\mathcal{F}$ *is a flat sheaf exactly when the following conditions hold*

(1) $\mathcal{F}$ *is a local functor (i.e. a Zariski sheaf)*
(2) *For* $X' \longrightarrow X$ *faithfully flat,* X', X *affine, the sequence*
$$\mathcal{F}(X) \to \mathcal{F}(X') \rightrightarrows \mathcal{F}(X' \underset{X}{\times} X')$$
is exact.

Proof. As Ia, b belong to the covering families, these conditions are necessary.

Let's show they are sufficient. First, let us see that the second condition holds universally: for any base change $Y \longrightarrow X$ we also have
$$\mathcal{F}(Y) \to \mathcal{F}(Y') \rightrightarrows \mathcal{F}(Y' \underset{Y}{\times} Y')$$

Let $\mathfrak{V} = (V_\alpha \subset Y)$ be an open affine covering of Y, let $\mathfrak{V}' = (V'_\alpha \to Y')$ be defined by base change, and $\mathfrak{V}''$ analogously ($X'' = X' \underset{X}{\times} X'$ etc.). V'_α is affine (as $\simeq X' \underset{X}{\times} V_\alpha$), as well as V''_α, so we have open affine coverings on Y, Y' and Y'':

$$\begin{array}{ccccc} X'' & \rightrightarrows & X' & \longrightarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ Y'' & \rightrightarrows & Y' & \longrightarrow & Y \\ \uparrow & & \uparrow & & \uparrow \\ V''_\alpha & \rightrightarrows & V'_\alpha & \longrightarrow & V_\alpha \end{array}$$

We have to do some chasing around in the next diagram

$$\begin{array}{ccccc} \mathcal{F}(Y) & \longrightarrow & \mathcal{F}(Y') & \rightrightarrows & \mathcal{F}(Y'') \\ \downarrow & & \downarrow & & \downarrow \\ \prod \mathcal{F}(V_\alpha) & \longrightarrow & \prod \mathcal{F}(V'_\alpha) & \rightrightarrows & \prod \mathcal{F}(V''_\alpha) \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \prod \mathcal{F}(V_{\alpha\beta}) & \longrightarrow & \prod \mathcal{F}(V'_{\alpha\beta}) & \rightrightarrows & \prod \mathcal{F}(V''_{\alpha\beta}) \end{array}$$

where $V_{\alpha\beta} = V_\alpha \cap V_\beta$, $V'_{\alpha\beta} = \dots$ etc. The columns are exact by the first assumption ($\mathcal{F}$ is local) and the middle row is exact by the second assumption. The injectivity $\mathcal{F}(Y) \hookrightarrow \mathcal{F}(Y')$ is clear, it follows from this $\mathcal{F}(V_{\alpha\beta}) \hookrightarrow \mathcal{F}(V'_{\alpha\beta})$. Chasing some more around proves the exactness of the first line.

Second, let us show the exactness of

$$\mathcal{F}(X) \longrightarrow \prod_\alpha \mathcal{F}(X_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(X_\alpha \underset{X}{\times} X_\beta)$$

for $(X_\alpha \to X)$ of type II, $X_\alpha \to U_\alpha \subset X$.

As $\mathcal{F}(X) \to \prod_\alpha \mathcal{F}(U_\alpha)$ is injective (as $\mathcal{F}$ is local) and $\mathcal{F}(U_\alpha) \to \mathcal{F}(X_\alpha)$ is injective (by the second assumption), we get $\mathcal{F}(X) \to \prod \mathcal{F}(X_\alpha)$ is injective as well.

Now let $s_\alpha \in \mathcal{F}(X_\alpha)$ be given such that $s_\alpha | X_\alpha \underset{X}{\times} X_\beta = s_\beta | X_\alpha \underset{X}{\times} X_\beta$.

We consider the following diagram with exact rows

$$\begin{array}{ccccc} \mathcal{F}(U_\alpha) & \longrightarrow & \mathcal{F}(X_\alpha) & \rightrightarrows & \mathcal{F}(X_\alpha \times_X X_\alpha) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(U_\alpha \cap U_\beta) & \longrightarrow & \mathcal{F}(X_\alpha \times_X U_\beta) & \rightrightarrows & \mathcal{F}(X_\alpha \times_X X_\alpha \times_X U_\beta) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(U_\alpha \times_X X_\beta) & \longrightarrow & \mathcal{F}(X_\alpha \times_X X_\beta) & \rightrightarrows & \mathcal{F}(X_\alpha \times_X X_\alpha \times_X X_\beta) \end{array}$$

and we see: $\forall \alpha \, \exists \, t^\alpha_\beta \in \mathcal{F}(U_\alpha \cap U_\beta)$ with $t^\alpha_\beta | X_\alpha \underset{X}{\times} U_\beta = s_\alpha | X_\alpha \underset{X}{\times} U_\beta$

Now look at (α fixed)

$$\begin{array}{ccc}
\mathcal{F}(U_\alpha) & \hookrightarrow & \mathcal{F}(X_\alpha) \\
\downarrow & & \downarrow \\
\prod_\beta \mathcal{F}(U_\alpha \cap U_\beta) & \hookrightarrow & \prod_\beta \mathcal{F}(X_\alpha \times_X U_\beta) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\prod_{\beta,\gamma} \mathcal{F}(U_\alpha \cap U_\beta \cap U_\gamma) & \hookrightarrow & \prod_{\beta,\gamma} \mathcal{F}(X_\alpha \times_X U_\beta \times_X U_\gamma)
\end{array}$$

with exact columns: $t^\alpha_\beta | U_\alpha \cap U_\beta \cap U_\gamma = t^\alpha_\gamma | U_\alpha \cap U_\beta \cap U_\gamma$
$\Rightarrow \exists\, t^\alpha \in \mathcal{F}(U_\alpha)$ with $t^\alpha | X_\alpha = s_\alpha$, $t^\alpha | U_\alpha \cap U_\beta = t^\alpha_\beta$
$\Rightarrow t^\alpha | U_\alpha \cap U_\beta = t^\beta | U_\alpha \cap U_\beta$, because both restricted to $X_\alpha \underset{X}{\times} X_\beta$ are equal
$\Rightarrow \exists\, t \in \mathcal{F}(X)$ with $t | U_\alpha = t^\alpha$, $\Rightarrow t | X_\alpha = s_\alpha$

We now conclude the proof of the theorem. Let $R \in J_{flat}(X)$, then $\exists \mathfrak{U}$ of type II with $R \supset R_\mathfrak{U}$.

$$\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{\sim} & \mathcal{F}(R_\mathfrak{U}) \\
\downarrow & \nearrow & \\
\mathcal{F}(R) & &
\end{array}$$

by what has just been shown.

We will now prove that $\mathcal{F}(R) \to \mathcal{F}(R_\mathfrak{U})$ is injective as well:

$$\begin{array}{ccccc}
R_\mathfrak{U} & \hookrightarrow & R & \overset{\varphi}{\underset{\psi}{\rightrightarrows}} & \mathcal{F} \\
\uparrow & & \uparrow{\scriptstyle y} & & \\
R_\mathfrak{U} \times_X Y & \hookrightarrow & Y & &
\end{array}$$

Take $\varphi, \psi \in \mathcal{F}(R)$ such that $\varphi | R_\mathfrak{U} = \psi | R_\mathfrak{U}$. To show $\varphi = \psi$, take any $Y, y \in R(Y)$ and $\varphi \circ y = \psi \circ y$ follows from the injectivity we already know $\mathcal{F}(Y) \hookrightarrow \mathcal{F}(R_\mathfrak{U} \underset{X}{\times} Y)$.

$\Rightarrow \mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(R)$, and $\mathcal{F}$ is a flat sheaf and theorem 3.8 is proven. □

The following theorem was announced in 3.1 as Theorem 3.2.

Theorem 3.9. *Any scheme is a flat sheaf.*

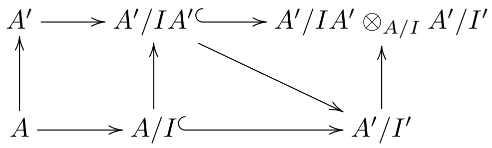
Proof. We apply theorem 3.8 to $\mathcal{F} = \mathrm{Hom}(\ ,X)$

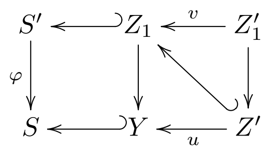
(1) $\mathrm{Hom}(\ ,X)$ is a local functor: continuous maps as well as sections of the structure sheaf glue together, because structure sheaves are sheaves in the Zariski topology.
(2) It remains to be shown exactness of $\mathrm{Hom}(S,X) \to \mathrm{Hom}(S',X) \rightrightarrows \mathrm{Hom}(S'',X)$ for $S = \mathrm{Spec}\, A$, $S' = \mathrm{Spec}\, A'$, $S'' = \mathrm{Spec}\, A''$, $A'' = A' \otimes_A A'$, A' faithfully flat over A.
   (a) $S'' \overset{p_1}{\underset{p_2}{\rightrightarrows}} S' \xrightarrow{\varphi} S$ is exact in the category of sets.

$$\begin{array}{ccccc}
S'' & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & S' & \xrightarrow{\varphi} & S \\
& {\scriptstyle h''}\searrow & \downarrow{\scriptstyle h'} & \swarrow{\scriptstyle h} & \\
& & Z & &
\end{array}$$

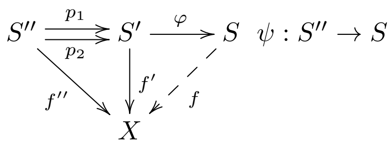
   Let $s \in S$ be a point, $s'_1, s'_2 \in S'$ be two preimages ($s = \varphi(s'_1) = \varphi(s'_2)$). Then $\exists\, s'' \in S''$ $p_1(s'') = s'_1$, $p_2(s'') = s'_2$ by the fact that S'' is the fibre product (see [8, Cor. 3.2.7.1]). As $h'(s'_1) = h''(s'') = h'(s'_2)$ the definition $h(s) := h'(s'_1)$ is well defined.

(b) S has the quotient topology of S': $Z \subset S$ is closed $\Leftrightarrow \varphi^{-1}(Z) \subset S'$ is closed
"$\Leftarrow$" this direction has to be shown
Let $Z' = \varphi^{-1}(Z) = V(I')$ for an ideal $I' \subset A'$. Put $I = A \cap I'$ and consider the diagram



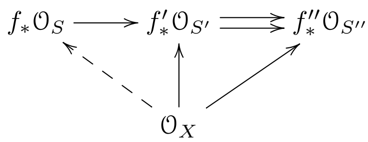


$Z_1 = \varphi^{-1}(Y)$, $Z' = \varphi^{-1}(Z) \subset Z_1$. We have $Z = \varphi(Z')$ and as u is dominant: $\overline{Z} = Y$.
$\varphi v(Z'_1) \subset u(Z') = Z, \Rightarrow\ v(Z'_1) \subset \varphi^{-1}(Z) = Z', \Rightarrow\ \overline{v(Z'_1)} = Z_1 \subset Z'$, therefore $Z_1 = Z'$
$\Rightarrow\ \varphi(Z') = Z = \varphi(Z_1) = Y$ and Z is closed.

(c) 

By 2a there is a unique application f, which by 2b is continuous. We have to see that a similar assertion holds for the structure sheaves.

Claim. $\mathcal{O}_S \to \varphi_*\mathcal{O}_{S'} \rightrightarrows \psi_*\mathcal{O}_{S''}$ is exact

therefore (f_* is left exact)  qed.

It remains to prove the claim: $\mathcal{O}_S \to \varphi_*\mathcal{O}_{S'} \rightrightarrows \psi_*\mathcal{O}_{S''}$ to be exact.
As the situation is analogous after localisation (over $S = \operatorname{Spec} A$), it suffices to prove the assertion for global sections:

$$A \xrightarrow{\varepsilon} A' \rightrightarrows A'' \text{ exact}$$

or to show exactness of

$$0 \to A \xrightarrow{\varepsilon} A' \xrightarrow{d} A''$$

where $da' = a' \otimes 1 - 1 \otimes a'$ (the difference of the two $A' \rightrightarrows A''$).
As $A \to A'$ is faithfully flat, it suffices to show this after tensoring $? \otimes A'$

$$0 \to A' \xrightarrow{\varepsilon'} A'' \xrightarrow{d'} A'''$$

where $\varepsilon'(a') = 1 \otimes a'$, $d'(a'_1 \otimes a'_2) = a'_1 \otimes 1 \otimes a'_2 - 1 \otimes a'_1 \otimes a'_2$ and to this end one constructs a *homotopy* for this complex as follows.
Let $m : A'' \to A'$ be the multiplication $m(a'_1 \otimes a'_2) = a'_1 \cdot a'_2$ and define the homotopy $s : A''' \longrightarrow A''$ by $s(a'_1 \otimes a'_2 \otimes a'_3) = a'_1 \otimes a'_2 a'_3$.
Now $m \circ \varepsilon' = id$ and $s \circ d' + \varepsilon' \circ m = id$, from which exactness ensues.

□

Theorem 3.10. *Let $\mathcal{F} \in S_{flat}$ be a flat sheaf $/S$.*

If $\mathcal{F}$ is locally affine representable, then it is globally affine representable.

Proof. As this is clear for open coverings, it is sufficent to assume S affine and that there is a $S' \xrightarrow{p} S$ of type Ib (see 3.6), such that $\mathcal{F}$ is affine representable over S'. We have a cartesian diagram (in the topos S_{flat}, or in the category of pre–sheaves, if you like)

$$\begin{array}{ccccccc} \mathcal{F} & \longleftarrow & \mathcal{F}' & \Leftarrow & \mathcal{F}'' & \Lleftarrow & \mathcal{F}''' \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f''}\downarrow & & {\scriptstyle f'''}\downarrow \\ S & \longleftarrow & S' & \Leftarrow & S'' & \Lleftarrow & S''' \end{array}$$

We have seen (in the proof of theorem 3.9), that $S \leftarrow S'$ is the Coker of $S' \leftleftarrows S''$ in the category of schemes, therefore also in S_{flat}. Hence the epimorphism $\mathcal{F} \leftarrow \mathcal{F}'$ is a Coker of $\mathcal{F}' \leftleftarrows \mathcal{F}''$ in the category of scheaves S_{flat}.

By assumption the sheaves $\mathcal{F}', \mathcal{F}'', \mathcal{F}'''$ are representable by affine schemes $\operatorname{Spec} B'$, $\operatorname{Spec} B''$, $\operatorname{Spec} B'''$. In the category of rings we get a diagram

$$\begin{array}{ccccccc} & & B' & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & B'' & \overset{\pi_{12}}{\Rrightarrow} & B''' \\ & & {\scriptstyle \varphi'}\uparrow & & {\scriptstyle \varphi''}\uparrow & & \uparrow{\scriptstyle \varphi'''} \\ A & \underset{i}{\hookrightarrow} & A' & \overset{i_1}{\underset{i_2}{\rightrightarrows}} & A'' & \overset{i_{12}}{\Rrightarrow} & A''' \end{array}$$

where A' is faithfully flat over A, $A'' = A' \otimes_A A'$, $A''' = A' \otimes_A A' \otimes_A A'$.

Define $B = \operatorname{Ker}(\pi_1, \pi_2) = \{b \in B' | \pi_1(b) = \pi_2(b)\}$.

We have to show that $B' \simeq B \otimes_A A'$.

$$\begin{array}{ccc} B & \overset{\pi}{\hookrightarrow} & B' \\ {\scriptstyle \varphi}\uparrow & & \uparrow{\scriptstyle \varphi'} \\ A & \overset{i}{\hookrightarrow} & A' \end{array}$$

$$\begin{array}{ccc} \mu : B \otimes_A A' & \longrightarrow & B' \\ \mu(b \otimes a') & = & \pi(b).\varphi'(a') \end{array}$$

Applying to

$$\begin{array}{ccccccc} & & B' & \xrightarrow{\pi_2} & B'' & \xrightarrow{\pi_{23}} & B''' \\ & & {\scriptstyle \varphi'}\uparrow & & {\scriptstyle \varphi''}\uparrow & & \uparrow{\scriptstyle \varphi'''} \\ A & \hookrightarrow & A' & \xrightarrow[i_2]{} & A'' & \xrightarrow[i_{23}]{} & A''' \end{array}$$

the (dual) reasoning of the remark 4.1 in section 4.1 (Base Change), we get

$$\mu' : B' \otimes_A A' \xrightarrow{\sim} B''$$
$$\mu'' : B'' \otimes_A A' \xrightarrow{\sim} B'''$$

given by $\mu'(b' \otimes a') = \pi_2(b').\varphi''(a' \otimes 1)$ and $\mu''(b'' \otimes a') = \pi_{23}(b'').\varphi'''(a' \otimes 1 \otimes 1)$.

Look at the diagram

$$\begin{array}{ccccc} B \otimes_A A' & \xrightarrow{\pi\otimes 1} & B' \otimes_A A' & \overset{\pi_1\otimes 1}{\underset{\pi_2\otimes 1}{\rightrightarrows}} & B'' \otimes_A A' \\ \mu \downarrow & & \mu' \downarrow & & \downarrow \mu'' \\ B' & \xrightarrow[\pi_1]{} & B'' & \overset{\pi_{12}}{\underset{\pi_{13}}{\rightrightarrows}} & B''' \end{array}$$

It has exact rows and is commutative, therefore μ an isomorphism. Hence π is faithfully flat and therefore $\operatorname{Spec} B \leftarrow \operatorname{Spec} B'$ a Coker of $\operatorname{Spec} B' \leftleftarrows \operatorname{Spec} B''$, that is $\mathcal{F} \simeq \operatorname{Spec} B$. □

4. Flat Descent

4.1. **Base Change.** To a $p : S' \to S$ construct the diagram

$$S \xleftarrow{p} S' \overset{p_1}{\underset{p_2}{\leftleftarrows}} S'' \overset{p_{12}}{\underset{p_{23}}{\overset{p_{13}}{\Lleftarrow}}} S'''$$

where $S'' = S' \underset{S}{\times} S'$, $S''' = S' \underset{S}{\times} S' \underset{S}{\times} S'$ and the morphisms are the canonical ones.

Moreover the following relations hold for the projections $q = pp_1 = pp_2 : S'' \longrightarrow S$, and $q_1, q_2, q_3 : S''' \Rrightarrow S'$

$$\begin{aligned} q_1 &= p_1 p_{12} = p_1 p_{13} \\ q_2 &= p_1 p_{23} = p_2 p_{12} \\ q_3 &= p_2 p_{23} = p_2 p_{13} \end{aligned}$$

For $f : X \longrightarrow S$ there is a cartesian diagram

$$\begin{array}{ccccccc} X & \longleftarrow & X' & \leftleftarrows & X'' & \Lleftarrow & X''' \\ f\downarrow & & f'\downarrow & & f''\downarrow & & f'''\downarrow \\ S & \longleftarrow & S' & \leftleftarrows & S'' & \Lleftarrow & S''' \end{array}$$

and the top level morphisms will be denoted similarly.

But when we are only given $X' \longrightarrow S'$, there are two fiber products with S'' and three with S''', $X''_1 = p_1^* X'$, $X''_2 = p_2^* X'$, $X'''_1 = q_1^* X'$, $\ldots$. The above diagram then becomes

$$\begin{array}{ccccccc} & & X' & \leftleftarrows & X''_1, X''_2 & \Lleftarrow & X'''_1, X'''_2, X'''_3 \\ & & \downarrow & & \downdownarrows & & \downarrow\downarrow\downarrow \\ S & \longleftarrow & S' & \leftleftarrows & S'' & \Lleftarrow & S''' \end{array}$$

The theory of descent describes the obvious glue conditions and undertakes to get descent data to recover an effective X/S.

Remark. As a final remark let us consider the cartesian diagram

$$\begin{array}{ccccc} X' & \xleftarrow{p_2} & X''_2 & \xleftarrow{p_{23}} & X'''_3 \\ f'\downarrow & & f''\downarrow & & f'''\downarrow \\ S' & \xleftarrow[p_2]{} & S'' & \xleftarrow[p_{23}]{} & S''' \end{array}$$

Then the following are cartesian as well

$$\begin{array}{ccc} X' & \xleftarrow{p_2} & X_2'' \\ {\scriptstyle pf'}\downarrow & & \downarrow{\scriptstyle p_1 f''} \\ S & \xleftarrow[p]{} & S' \end{array}$$

$$\begin{array}{ccc} X_2'' & \xleftarrow{p_{23}} & X_3''' \\ {\scriptstyle qf''}\downarrow & & \downarrow{\scriptstyle q_1 f'''} \\ S & \xleftarrow[p]{} & S' \end{array}$$

This is obvious for sets, and thus sufficient by general nonsense.

4.2. **Glue Condition and Descent Data.** In this section we will consider a morphism $p : S' \longrightarrow S$ as well as the derived diagrams from base change.

One wants to describe the category of schemes over S by the schemes over S' equipped with an additional structure.

Let $f' : X' \to S'$ be given and let $f_1'' : X_1'' \longrightarrow S''$, $f_2'' : X_2'' \longrightarrow S''$ be the two reverse images under $S'' \rightrightarrows S$.

Definition 4.1. An S''–isomorphism $u : X_2'' \longrightarrow X_1''$ is called a *glue datum* on X' relative to p and a *descent datum* if additionally for the pull backs to S''' of this isomorphism

$$u_{12}. = p_{12}^*(u) : X_2''' \xrightarrow{\sim} X_1'''$$
$$u_{13}. = p_{13}^*(u) : X_3''' \xrightarrow{\sim} X_1'''$$
$$u_{23}. = p_{23}^*(u) : X_3''' \xrightarrow{\sim} X_2'''$$

the following relation holds

$$u_{13} = u_{12} \cdot u_{23}$$

If X' is of the form $X \underset{S}{\times} S'$ for a $f : X \longrightarrow S$, there is a canonical datum of descent. Such data of descent are called *effective*.

Let (X', u), (Y', v) be glue data. A morphism $h : (X', u) \longrightarrow (Y', v)$ is an S'–morphism $h : X' \longrightarrow Y' /S'$ which is compatible with u, v: $vh_2 = h_1 u$

$$\begin{array}{ccc} X_2'' & \xrightarrow{h_2} & Y_2'' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X_1'' & \xrightarrow{h_1} & Y_1'' \end{array}$$

We have thus defined a category $\mathcal{G}lue(S'/S)$ of glue data, together with a full subcategory $\mathcal{D}esc(S'/S)$ of descent data, as well as a functor

$$\begin{array}{rcl} \Delta : Sch/S & \longrightarrow & \mathcal{D}esc(S'/S) \\ X/S & \longmapsto & (X \underset{S}{\times} S', u_{can}) \end{array}$$

onto the subcategroy of effective descent data.

Definition 4.2. The morphism $p : S' \longrightarrow S$ is called a *descent morphism*, iff Δ is fully faithful, and to be an *effective descent morphism*, iff it is an equivalence of categories.

Theorem 4.1. *A faithfully flat and quasi–compact morphism $p : S' \longrightarrow S$ is a descent morphism.*

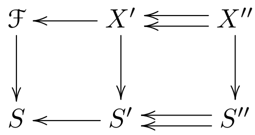
Proof. This is a corollary to theorem 3.9 (or 3.1): the sequence

$$\mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_{S'}(X',Y') \rightrightarrows \mathrm{Hom}_{S''}(X'',Y'')$$

is exact. □

Theorem 4.2. *Any affine descent datum with respect to a faithfully flat and quasi–compact morphism is effective.*

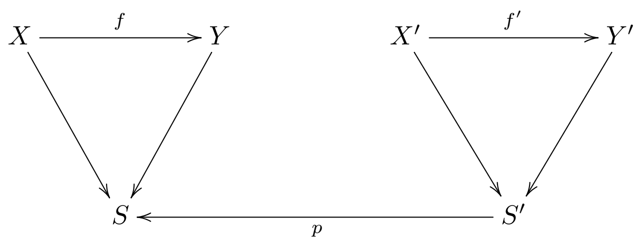
Proof. Corollary to theorem 3.10: you consider the cokernel



in the category of sheaves (where via $u : X''_2 \simeq X''_1 = .X''$).

The canonical arrow $X' \to \mathcal{F} \underset{S}{\times} S'$ is an isomorphism, because the upper sequence exhibits the sheaf $\mathcal{F} \underset{S}{\times} S'$ after base change $S' \to S$ as the cokernel of $X'' \overset{p_{12}}{\underset{p_{13}}{\leftleftarrows}} X'''$, which on the other hand has the cokernel X'. This means that $\mathcal{F}$ is locally affine representable and we can apply the representability theorem 3.10. □

4.3. **Descent of properties of morphisms.**



Most properties remain valid after descending from f' to f:

> **surjective, injective:** EGA [9, IV,2.6.1]
> **open, closed:** EGA [9, IV, 2.6.2]
> **quasi–compact:** EGA [9, IV, 2.6.3]
> **(quasi–)separated:** EGA [9, IV, 2.7.1]
> **(locally) finitely generated:** EGA [9, IV, 2.7.1]
> **(locally) finitely presented:** EGA [9, IV, 2.7.1]
> **open, closed immersion:** EGA [9, IV, 2.7.1]
> **proper:** EGA [9, IV, 2.7.1]
> **affine:** EGA [9, IV, 2.7.1]
> **integral, finite:** EGA [9, IV, 2.7.1]
> **flat:** 3b in section 3.2.2
> **smooth:** EGA [9, IV, 17.1]
> **unramified:** EGA [9, IV, 17.1]
> **étale:** EGA [9, IV, 17.1]

Not conserved is *projective*, see EGA [9, IV, 2.7.3] by a counter example of Hironaka.

4.4. **Descending quasi–coherent Modules.** How can quasi–coherent $\mathcal{O}_{S'}$–modules $\mathcal{M}'$ be characterized, which come from a $\mathcal{O}_S$–module $\mathcal{M}$: $\mathcal{M}' \simeq p^*\mathcal{M}$?

A *glue datum* of an $\mathcal{O}_{S'}$–module $\mathcal{M}'$ is an $\mathcal{O}_{S''}$–isomorphism $u : p_2^*\mathcal{M}' \xrightarrow{\sim} p_1^*\mathcal{M}'$, and it is a *descent datum*, if $u_{13} = u_{12}u_{23}$.

Theorem 4.3. *Let $p : S' \longrightarrow S$ be faithfully flat and quasi–compact. Any descent datum of quasi–coherent $\mathcal{O}_{S'}$–modules is effective.*

Proof. Define to $\mathcal{M}'$ the pre–sheaf $/S'$

$$\mathbf{V}(\mathcal{M}')(T') := \mathrm{Hom}_{\mathcal{O}_{S'}-Mod}(\mathcal{M}', \varphi_*\mathcal{O}_{T'}) \qquad \text{for } \varphi : T' \to S'$$

which is a locally affine representable flat sheaf, therefore $X' = \mathbf{V}(\mathcal{M}')$ is a scheme and

$$\begin{array}{c} \mathbf{V}(\mathcal{M}') \\ \big\downarrow \scriptstyle\text{affine} \\ S' \end{array}$$

is the so called *vector bundle* associated to $\mathcal{M}'$ over S'.

A descent datum for $\mathcal{M}'$ gives one for $\mathbf{V}(\mathcal{M}')$ and by theorem 4.2 this is effective: $\exists\, X/S$ affine with

$$\begin{array}{ccc} X & \xleftarrow{\varphi} & \mathbf{V}(\mathcal{M}') \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ S & \xleftarrow[p]{} & S' \end{array}$$

Now we have $p^*f_*\mathcal{O}_X \simeq f'_*\mathcal{O}_{X'} = \mathcal{O}_{S'}[\mathcal{M}'] = \mathcal{O}_{S'} \oplus \mathcal{M}' \oplus \cdots$ and this graduation induces one of $f_*\mathcal{O}_X$, let $\mathcal{M}$ be the dimension 1 component, $p^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ and $f_*\mathcal{O}_X = \mathcal{O}_S[\mathcal{M}]$, that is $X \simeq \mathbf{V}(\mathcal{M})$ is itself a vector bundle. □

Remark. A direct proof of theorem 4.3 (descent of quasi–coherent modules) occurs in the literature SGA [7, VIII] Theorems 4.1 and 4.2 are deduced from theorem 4.3 in loc.cit.

4.5. **Examples for descent.**

(1) Let k be a field, E a finite dimensional k–vector space. We consider the k–morphism

$$p : S' \longrightarrow S$$

where $S' = \mathbf{V}(E) - \{0\}$ and $S = \mathbf{P}(E)$.

In a basis $x_0, x_1, \ldots, x_n$ of E, p is (Zariski–) locally of the form

$$k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \hookrightarrow k[x_0, \ldots, x_n, \frac{1}{x_i}]$$

therefore affine, hence quasi–compact, furthermore free, hence flat. Finally p is surjective, so p is quasi–compact, faithfully flat.

On the fibers (lines) of p operates the multiplicative group $\mu/k = \mathrm{Spec}(k[\mathbb{Z}])$ by homotheties and

$$\begin{array}{ccc} \mu \times S' & \xrightarrow{\sim} & S' \underset{S}{\times} S' = S'' \\ \sigma, s' & \longmapsto & \sigma s', s' \end{array}$$

and by this we can identify

$$\begin{array}{ccc} \mu \times \mu \times S' & \xrightarrow{\sim} & S''' \\ (\sigma, \tau, s') & \longmapsto & (\sigma\tau s', \tau s', s') \end{array}$$

The canonical projections are

$$\begin{aligned} p_{12}(\sigma,\tau,s') &= (\sigma,\tau s') \\ p_{13}(\sigma,\tau,s') &= (\sigma\tau,s') \\ p_{23}(\sigma,\tau,s') &= (\tau,s') \end{aligned}$$

In the form $\mu \times S'$ a glue datum on a S'–scheme X' is given by a morphism

$$\begin{array}{ccc} \mu \times X' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \mu \times S' & \longrightarrow & S' \end{array}$$

with commutative diagram and this is a datum for descent exactly when this defines an operation of the multiplicative group μ on X'.

As a result: an affine scheme over $\mathbf{V}(E) - \{0\}$, on which μ operates compatibly, comes from a scheme affine over $\mathbf{P}(E)$.

(2) Let k'/k be a finite Galois extension with group G.
$S' = \operatorname{Spec} k' \longrightarrow S = \operatorname{Spec} k$ is faithfully flat, quasi–compact

$$\begin{array}{ccc} k' \otimes_k k' & \longrightarrow & k' \times \cdots \times k' \\ x' \otimes y' & \longmapsto & (\sigma(x')y')_{\sigma\in G} \end{array}$$

$S'' \simeq S' \times G$. As above we have: A datum for descent on X'/S' is given by a G–operation.

For example for $k = \mathbb{R}$, $k' = \mathbb{C}$ giving a descent datum on $X'/\mathbb{C}$ is equivalent to giving an isomorphism

$$\begin{array}{ccc} X' & \xrightarrow{\;u\;} & X' \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \xrightarrow[c]{} & \operatorname{Spec}\mathbb{C} \end{array}$$

where c is complex conjugation and $u^2 = id$.

If $X' = \operatorname{Spec}\mathbb{C}[T]/(T^2+1)$ with $u(T) = T$, then
$X = \operatorname{Spec}\mathbb{R}[T]/(T^2+1)$.

But for $u(T) = -T$, we find that $X = \operatorname{Spec}\mathbb{R}[W]/(W^2-1)$ with $W = iT$.

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