

LOCAL ε -FACTORS

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For Petra

ABSTRACT. Note on local L -series, functional equation, ε -factors, GAUSS sums and integrals.

1. LOCAL ζ -FUNCTIONS

1.1. Definition of L -function and Z -integral. K is a local field and $\varphi \in \mathcal{S}(K)$ a SCHWARTZ function (that is: \mathcal{C}^∞ with compact support; in the non-archimedean case this basically means a finite function).

Definition 1.1.

$$Z(\lambda, \varphi) = \int_{K^\times} \lambda(t) \varphi(t) d_1^\times t \quad L(\lambda) = \begin{cases} \frac{1}{1 - \lambda(\mathfrak{p})} & \lambda \in \Lambda_K, \lambda \neq \omega_0 \\ 1 & \lambda \in \Omega_K - \Lambda_K \end{cases}$$

Theorem 1.1. *The Z integrals converge for $\operatorname{Re} \lambda = \sigma > 0$, in the ramified case even for all $\lambda \in \Omega_K - \Lambda_K$. The function $\frac{Z(\lambda, \varphi)}{L(\lambda)}$ is holomorph, which gives a meromorphic continuation of the Z -integral on all of Ω_K .*

Functional equation¹

$$(1) \quad \frac{Z(\omega_1 \lambda^{-1}, \widehat{\varphi})}{L(\omega_1 \lambda^{-1})} = \varepsilon(\lambda, \tau) \cdot \frac{Z(\lambda, \varphi)}{L(\lambda)}$$

Moreover $Z(\lambda, \varphi)$ is holomorph for all $\lambda \in \Omega_K$ in case $\varphi(0) = 0$ (i.e. if $\varphi \in \mathcal{S}(K^\times)$).

1.2. Calculation of Z -function and the ε -factor. Let $\varphi \in \mathcal{S}(K)$, then $\exists \ell \leq k$ with $\varphi(x+y) = \varphi(x) \quad \forall y \in \mathfrak{p}^k$ and support $\operatorname{supp} \varphi \subset \mathfrak{p}^\ell$.

Let φ_k be the characteristic function of \mathfrak{p}^k , $\varphi = \sum \varphi(x) T_x \varphi_k$, where $(T_x \varphi)(y) = \varphi(y-x)$ and $x \in \mathfrak{p}^\ell$ runs through representatives modulo \mathfrak{p}^k . Without restriction $\varphi = T_x \varphi_k$ with $\operatorname{ord} x \leq k$.

Lemma 1.2.

$$\widehat{T_x \varphi_k} = \tau_x \cdot \varphi_{-k-\delta} \cdot q^{-k-\delta/2}$$

with $\delta = \operatorname{ord} \tau$ as in WEIL [1, def. 4 in II, §5].

Proof. By calculating

$$\widehat{T_x \varphi_k}(y) = \int_K \varphi_k(z-x) \tau(yz) dz = \tau(yx) \int_K \varphi_k(z) \tau(yz) dz = (\tau_x \cdot \widehat{\varphi_k})(y)$$

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¹for τ see section 2.1

and

$$\begin{aligned} (\tau_x \cdot \widehat{\varphi}_k)(y) &= \int_{\mathfrak{p}^k} \tau(yx) dx = \begin{cases} 0 & \tau_y | \mathfrak{p}^k \neq 1 \\ \text{vol}(\mathfrak{p}^k) & \tau_y | \mathfrak{p}^k = 1 \end{cases} \\ \tau_y | \mathfrak{p}^k = 1 &\iff y \in \mathfrak{p}^{-\delta-k} \\ \text{vol}_\tau(\mathfrak{p}^k) &= q^{-k} \cdot \text{vol}_\tau(\mathfrak{o}) = q^{-k-\delta/2} \end{aligned}$$

concluding the proof. \square

We are going to calculate the Z -integrals $Z(\lambda, T_x \varphi_k)$, $Z(\lambda, \tau_x \varphi_k)$ and the assertions of theorem 1.1 will arise out of them.

Let now $\text{Re } \lambda = \sigma > 0$.

$$\begin{aligned} Z(\lambda, \varphi_k) &= \int_{0 \neq |t| \leq q^{-k}} \lambda(t) d_1^\times t = \left(\sum_{n \geq k} \lambda(\pi^n) \right) \cdot \int_{\mathfrak{o}^\times} \lambda(t) d_1^\times t = \\ &= \frac{\lambda(\mathfrak{p})^k}{1 - \lambda(\mathfrak{p})} \quad \text{for } \lambda \in \Lambda_K, \\ &= 0 \quad \text{otherwise } \lambda \notin \Lambda_K. \end{aligned}$$

$$(2) \quad \frac{Z(\lambda, \varphi_k)}{L(\lambda)} = \lambda(\mathfrak{p})^k \quad \lambda \in \Lambda_K$$

Let now $x \in K^\times$ have $\text{ord } x = \ell < k$.

$$\begin{aligned} Z(\lambda, T_x \varphi_k) &= \int_{K^\times} \lambda(t) \varphi_k(t-x) d_1^\times t = \lambda(x) \int_{K^\times} \lambda(t) \varphi_k(x(t-1)) d_1^\times t = \\ &= \lambda(x) \int_{1+\mathfrak{p}^{k-\ell}} \lambda(t) d_1^\times t \end{aligned}$$

$$\text{as } \varphi_k(x(t-1)) \neq 0 \iff x(t-1) \in \mathfrak{p}^k \iff t-1 \in \mathfrak{p}^{k-\ell} \iff t \in 1 + \mathfrak{p}^{k-\ell}$$

Now $\lambda|1 + \mathfrak{p}^{k-\ell} = 1 \iff f(\lambda) \leq k - \ell$ so that we get

$$(3) \quad Z(\lambda, T_x \varphi_k) = \begin{cases} \lambda(x) \cdot q^{1+\ell-k} \cdot (q-1)^{-1} & \text{for } f(\lambda) \leq k - \ell \\ 0 & \text{otherwise } (f(\lambda) > k - \ell). \end{cases}$$

Under the same assumption

$$\begin{aligned} Z(\lambda, \tau_x \varphi_{-k-\delta}) &= \int_{K^\times} \lambda(t) \tau(xt) \varphi_{-k-\delta}(t) d_1^\times t = \\ &= \sum_{n \geq -k-\delta} \lambda(\pi^n) \int_{\mathfrak{o}^\times} \lambda(t) \tau(\pi^n xt) d_1^\times t = \sum_{n \geq -k-\delta} \lambda(\pi^n) \cdot \gamma(\lambda, \pi^n x) \end{aligned}$$

From prop. 2.1 we collect the results for $\lambda \in \Lambda_K$

$$\begin{aligned} \ell + n \geq -\delta & \quad \gamma(\lambda, \pi^n x) = 1 \\ \ell + n = -\delta - 1 & \quad \gamma(\lambda, \pi^n x) = \frac{1}{1-q} \\ \ell + n < -\delta - 1 & \quad \gamma(\lambda, \pi^n x) = 0 \end{aligned}$$

$$\begin{aligned} \text{For } \lambda \in \Lambda_K \quad Z(\lambda, \tau_x \varphi_{-k-\delta}) &= \lambda(\mathfrak{p})^{-1-\ell-\delta} \frac{1}{1-q} + \frac{\lambda(\mathfrak{p})^{-\ell-\delta}}{1-\lambda(\mathfrak{p})} = \\ &= \lambda(\mathfrak{p})^{-1-\ell-\delta} L(\lambda) \frac{1-q\lambda(\mathfrak{p})}{1-q} \end{aligned}$$

$$(4) \quad Z(\lambda, \tau_x \varphi_{-k-\delta}) = \lambda(\mathfrak{p})^{-1-\ell-\delta} \cdot \frac{L(\lambda) \cdot L(\omega_{-1})}{L(\omega_{-1}\lambda)}$$

For $\lambda \notin \Lambda_K$, $\gamma(\lambda, x\pi^n) \neq 0 \iff n + \ell = -\delta - f(\lambda)$. For $f(\lambda) > k - \ell$ is $n < -\delta - k$, this does not occur and $Z(\lambda, \tau_x \varphi_{-k-\delta}) = 0$. So, let $0 \neq f(\lambda) \leq k - \ell$, we have exactly one summand corresponding to $n = -\ell - f(\lambda) - \delta$:

$$Z(\lambda, \tau_x \varphi_{-k-\delta}) = \lambda(\pi)^{-\ell - f(\lambda) - \delta} \cdot \gamma(\lambda, x\pi^{-\delta - f(\lambda) - \ell})$$

and feeding the results in case $f(\lambda) > 0$ from prop. 2.1

$$\begin{aligned} (5) \quad Z(\lambda, \tau_x \varphi_{-k-\delta}) &= \lambda(\pi)^{-\ell - f(\lambda) - \delta} \cdot \frac{q^{-f(\lambda)/2}}{1 - q^{-1}} \cdot \bar{\kappa} \cdot \overline{\lambda(-x\pi^{-\ell})} = \\ &= \lambda(\pi)^{-\delta - f(\lambda)} \lambda(-1) \lambda(x)^{-1} \bar{\kappa} \cdot \frac{q^{-f(\lambda)/2}}{1 - q^{-1}} \end{aligned}$$

where $\bar{\kappa} = \lambda(-1)q^{-f(\lambda)/2}G(\lambda, \pi^{\delta+f(\lambda)})$ is a normalized GAUSS sum, see section 2.

We are now ready to *verify the functional equation* (1) in the two cases $\varphi = \varphi_k$ and $\varphi = T_x \varphi_k$ with $\text{ord } x = \ell < k$.

First the case $\varphi = \varphi_k$. For $\lambda \notin \Lambda_K$ both sides of (1) are zero. For $\lambda \in \Lambda_K$ we get by (2)

$$\frac{Z(\omega_1 \lambda^{-1}, \widehat{\varphi}_k)}{L(\omega_1 \lambda^{-1})} = q^{-k-\delta/2} \cdot \omega_1(\mathfrak{p})^{-k-\delta} \lambda(\mathfrak{p})^{k+\delta} = q^{\delta/2} \cdot \lambda(\mathfrak{p})^\delta \cdot \frac{Z(\lambda, \varphi_k)}{L(\lambda)}$$

verifying (1) with the local ε -factor

$$(6) \quad \varepsilon(\lambda, \tau) = (\omega_{-1/2} \lambda)(\mathfrak{p})^\delta, \quad \lambda \in \Lambda_K$$

Next the case $\varphi = T_x \varphi_k$ with $\text{ord } x = \ell < k$. The functional equation is again valid for $f(\lambda) > k - \ell$, as both sides vanish. For $\lambda \in \Lambda_K$, by Lemma 1.2 and (4)

$$\begin{aligned} \frac{Z(\omega_1 \lambda^{-1}, \widehat{\varphi})}{L(\omega_1 \lambda^{-1})} &= q^{-k-\delta/2} \cdot (\omega_1 \lambda^{-1})(\mathfrak{p})^{-1-\ell-\delta} \cdot \frac{L(\omega_{-1})}{L(\lambda^{-1})} = \\ &= q^{-k-\delta/2} \cdot q^{1+\ell+\delta} \cdot \lambda(\mathfrak{p})^{1+\ell+\delta} \cdot \frac{1 - \lambda^{-1}(\mathfrak{p})}{1 - q} = \end{aligned}$$

as $\lambda(x) = \lambda(\mathfrak{p})^\ell$

$$= q^{\delta/2} \cdot \lambda(\mathfrak{p})^\delta \cdot \lambda(x) \cdot q^{1+\ell-k} \cdot \frac{1 - \lambda(\mathfrak{p})}{q - 1} =$$

by (6) and (3)

$$= \varepsilon(\lambda, \tau) \cdot \frac{Z(\lambda, \varphi)}{L(\lambda)}$$

Now let λ be ramified with $f(\lambda) \leq k - \ell$, by Lemma 1.2 and (5)

$$\begin{aligned} Z(\omega_1 \lambda^{-1}, \widehat{\varphi}) &= q^{-k-\delta/2} Z(\omega_1 \lambda^{-1}, \tau_x \varphi_{-k-\delta}) = \\ &= q^{-k-\delta/2} q^{\delta+f(\lambda)} \lambda(\pi)^{\delta+f(\lambda)} \lambda(-1) |x|^{-1} \lambda(x) \bar{\kappa}(\lambda^{-1}) \cdot \frac{q^{-f(\lambda)/2}}{1 - q^{-1}} = \\ &= q^{\delta/2+f(\lambda)/2} \cdot \lambda(\pi)^{\delta+f(\lambda)} \lambda(-1) \bar{\kappa}(\lambda^{-1}) \cdot q^{1+\ell-k} \lambda(x) \cdot (q - 1)^{-1} \end{aligned}$$

as $\lambda(-1) \bar{\kappa}(\lambda^{-1}) = \kappa(\lambda)$ and using (3)

$$= (\omega_{-1/2} \lambda)(\pi)^{\delta+f(\lambda)} \cdot \kappa(\lambda) \cdot Z(\lambda, \varphi)$$

which proves (1) with the local ε -factor in the ramified case

$$(7) \quad \varepsilon(\lambda, \tau) = (\omega_{-1/2} \lambda)(\pi)^\delta \cdot \kappa(\lambda), \quad \lambda \notin \Lambda_K$$

2. GAUSS INTEGRALS AND GAUSS SUMS

2.1. Definition. Let $\tau : K \rightarrow \mathbf{S}^1$ be a fixed character and $d\tau$ the self dual HAAR measure on K relative to τ . Let $\delta = \text{ord } \tau$ be the order and $d \in K^\times$ such that $\text{ord } d = \delta$, see WEIL [1, def. 4 in II, §5].

Let $\omega \in \Omega(K)$ be of conductor \mathfrak{p}^f , see WEIL [1, def. 7 in VII, §3], write its exponent $f = f(\omega)$, let $c \in K^\times$ be such that $\text{ord } c = f$, set $b = c \cdot d$.

The GAUSS sum is defined by

Definition 2.1.

$$G(\omega, b) = \sum_{\varepsilon \in \mathfrak{o}^\times / 1 + \mathfrak{p}^f} \omega(\varepsilon) \tau(b^{-1} \varepsilon)$$

This is well defined as for $y \in \mathfrak{p}^f = c \cdot \mathfrak{o}$ we have $b^{-1}y \in d^{-1}\mathfrak{o}$, hence $\tau(b^{-1}(c + y)) = \tau(b^{-1}c)$ and $\omega(c + y) = \omega(c) \cdot \omega(1 + c^{-1}y) = \omega(c)$

The GAUSS integral is defined by

Definition 2.2.

$$\gamma(\omega, x) = \int_{\mathfrak{o}^\times} \omega(t) \tau(tx) d_1^\times t$$

where $d_1^\times t$ is normed to $\text{vol}(\mathfrak{o}^\times) = 1$.

We define $\varphi_\omega : K \rightarrow \mathbf{S}^1$ by $\varphi_\omega|_{\mathfrak{o}^\times} = \omega|_{\mathfrak{o}^\times}$ and $= 0$ elsewhere.

Remark. Formally $\gamma(\omega, x)$ is the multiplicative FOURIER coefficient of τ_x .

Proposition 2.1. *We have*

$$\begin{aligned} f = 0 & \quad \gamma(\omega, x) = \begin{cases} 1 & \text{ord } x \geq -\delta \\ \frac{-1}{q-1} & \text{ord } x = -\delta - 1 \\ 0 & \text{ord } x < -\delta - 1 \end{cases} \\ f > 0 & \quad \gamma(\omega, x) = \frac{|c|^{1/2}}{1 - q^{-1}} \cdot \kappa^{-1} \cdot \overline{\varphi_\omega(-bx)} \end{aligned}$$

with $\kappa = |b|^{-1/2} \cdot \widehat{\varphi_\omega}(b^{-1}) = \omega(-1) |c|^{1/2} \overline{G(\omega, b)}$ a normalized GAUSS sum, $\kappa \in \mathbf{S}^1$, see WEIL [1, VII, §7, prop. 13].

2.2. Calculation. The proof proceeds by playing back the calculation to the additive FOURIER transform and utilising self duality. Of course the unramified case is clear. Thus let $f \geq 1$. As is well-known $d_1^\times t = \frac{|d|^{-1/2}}{1 - q^{-1}} \cdot \frac{d\tau(t)}{|t|}$. Hence

$$\gamma(\omega, x) = \frac{|d|^{-1/2}}{1 - q^{-1}} \cdot \widehat{\varphi_\omega}(x)$$

As the support of φ_ω is $\mathfrak{o}^\times \subset \mathfrak{o}$ and φ_ω is constant on classes modulo $\mathfrak{p}^f = c \cdot \mathfrak{o}$, the support of $\widehat{\varphi_\omega}$ is in $\mathfrak{p}^{-\delta-f} = b^{-1} \cdot \mathfrak{o}$ and constant modulo $d^{-1} \cdot \mathfrak{o}$.

Assertion 2.1.

$$\text{supp } \widehat{\varphi_\omega} = b^{-1} \mathfrak{o}^\times$$

Proof. Let $x \in b^{-1}\mathfrak{p}$, we will show $\widehat{\varphi_\omega}(x) = 0$. If $f = 1$, $\widehat{\varphi_\omega}(x) = \int_{\mathfrak{o}^\times} \omega(t) d\tau(t) = 0$, as ω is non-trivial.

For $f > 1$ let $R \subset \mathfrak{o}^\times$ be a system of representatives $\mathfrak{o}^\times = R(1 + \mathfrak{p}^{f-1})$:

$$\widehat{\varphi}_\omega(x) = \sum_{\rho \in R} \int_{\mathfrak{p}^{f-1}} \varphi_\omega(\rho + t) \tau((\rho + t)x) d\tau(t) = \sum_{\rho \in R} \omega(\rho) \tau(\rho x) \int_{\mathfrak{p}^{f-1}} \omega(1 + t) d\tau(t)$$

which again = 0 as the last integral vanishes. \square

Now $\widehat{\varphi}_\omega(x\varepsilon) = \omega(\varepsilon^{-1})\widehat{\varphi}_\omega(x)$ for $\varepsilon \in \mathfrak{o}^\times$, $\implies \widehat{\varphi}_\omega(x) = \widehat{\varphi}_\omega(b^{-1})\overline{\widehat{\varphi}_\omega(bx)} \quad \forall x \in K$.

By development we get $\widehat{\varphi}_\omega(b^{-1}) = \sum_{\mathfrak{o}^\times \bmod \mathfrak{p}^f} \omega(\varepsilon) \tau(\varepsilon b^{-1}) \cdot \text{vol}_{d\tau}(\mathfrak{p}^f)$, therefore is $\widehat{\varphi}_\omega(b^{-1}) = G(\omega, b) \cdot |c| \cdot |d|^{1/2}$.

By FOURIER inversion follows

$$\begin{aligned} \varphi_\omega(t) &= \int_K \widehat{\varphi}_\omega(x) \tau(-tx) d\tau(x) = \widehat{\varphi}_\omega(b^{-1}) \int_K \overline{\widehat{\varphi}_\omega(bx) \tau(tx)} d\tau(x) = \\ &= \widehat{\varphi}_\omega(b^{-1}) \cdot |b|^{-1} \int_K \overline{\varphi_\omega(x) \tau(b^{-1}tx)} d\tau(x) = \\ &= \widehat{\varphi}_\omega(b^{-1}) \cdot |b|^{-1} \overline{\widehat{\varphi}_\omega(b^{-1}t)} = \\ &= |b|^{-1} \cdot \widehat{\varphi}_\omega(b^{-1}) \cdot \overline{\widehat{\varphi}_\omega(b^{-1})} \cdot \varphi_\omega(t) \end{aligned}$$

which already implies $\kappa \cdot \bar{\kappa} = 1$. As $\widehat{\varphi}_{\bar{\omega}}(x) = \omega(-1)\overline{\widehat{\varphi}_\omega(x)}$ the explicit formulas of prop. 2.1 follow.

REFERENCES

- [1] André Weil, *Basic Number Theory*, Classics in Mathematics, Springer, Berlin, Heidelberg, New York, 1973, 1995.