COMPACT OPERATORS

BERNDT E. SCHWERDTFEGER

Preface

This note is a *fact sheet* on compact operators with references rather than proofs given.

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Compact operators

We consider a normed space E with norm $\| \ \|$. We note $U(r) = \{x \in E \mid \|x\| < r\}$ the neighborhood of 0 of radius r.

Definition 1.1. Let E, F be normed spaces, $u : E \to F$ be linear. u is a *compact* operator, if the following equivalent properties are satisfied:

(1) The image of any bounded set under u is relatively compact.
(2) $u(U(1))$ is relatively compact.
(3) For a bounded sequence $(x_n)_n$, $x_n \in E$, the sequence $u(x_n)_n$ contains a convergent subsequence.

See [1, 11.2]. We denote the space of continuous linear operators by $\mathscr{L}(E, F)$ and the space of compact linear operators by $\mathscr{L}_c(E, F)$. As a compact operator is bounded, it is continuous: $\mathscr{L}_c(E, F) \subset \mathscr{L}(E, F)$.

Theorem 1.1. *Let $u : E \longrightarrow E$ be a compact operator, $v = 1 - u$. Then*

(1) *the kernel* $\operatorname{Ker} v = v^{-1}(0)$ *is finite dimensional;*
(2) *the image* $\operatorname{Im} v = v(E)$ *is closed in E;*
(3) *the cokernel* $\operatorname{Coker} v = E/v(E)$ *is finite dimensional;*
(4) *if v is injective, it is surjective and $v : E \xrightarrow{\sim} E$ is an automorphism.*

For $n \geq 1$ let $K_n = \operatorname{Ker} v^n$, $I_n = \operatorname{Im} v^n$, then the K_n form an increasing sequence of finite dimensional subspaces, and the I_n form a decreasing sequence of finite codimensional closed subspaces; put $K_\infty = \bigcup K_n$ and $I_\infty = \bigcap I_n$.

There is a smallest number k such that $K_{n+1} = K_n$ for $n \geq k$, so $K_\infty = K_k$. We also have $I_{n+1} = I_n$ for $n \geq k$, $I_\infty = I_k$, $E = K_\infty \oplus I_\infty$ and $v|I_\infty : I_\infty \xrightarrow{\sim} I_\infty$.

Proof. See Dieudonné [1, 11.3.2-4]. □

Theorem 1.1 is due to Frigyes Riesz.

Theorem 1.2. *Let $u : E \to E$ be a compact operator on a complex normed space. Then*

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(1) *the spectrum* S *of* u *is a denumerable compact subset of* $\mathbf{C}$, *each point* $\lambda \neq 0$ *is isolated and* $0 \in S$ *if* E *is infinite dimensional.*
(2) $S - \{0\}$ *consists of eigenvalues of* u.
(3) *For each* $\lambda \in S - \{0\}$ *there is a decomposition* $E = K(\lambda) \oplus I(\lambda)$ *with* $K(\lambda)$ *finite dimensional and* $I(\lambda)$ *closed such that*
   - $u(K(\lambda)) \subset K(\lambda)$ *and there is a smallest integer* $k = k(\lambda)$ *such that* $(u - \lambda)^k | K(\lambda) = 0$.
   - $u(I(\lambda)) \subset I(\lambda)$ *and* $u - \lambda | I(\lambda)$ *is an automorphism.*
(4) *The eigenspace* $E(\lambda)$ *for an eigenvalue* $\lambda \in S - \{0\}$ *satisfies* $E(\lambda) \subset K(\lambda)$, *hence is finite dimensional.*
(5) *for two eigenvalues* $\lambda \neq \mu$ $K(\mu) \subset I(\lambda)$
(6) *If* E *is complete (i.e. a* Banach *space), the analytic function*

$$\mathbf{C} - S \longrightarrow \mathscr{L}(E)$$
$$s \longmapsto (u - s)^{-1}$$

has a pole of order $k(\lambda)$ *at each* $\lambda \neq 0$ *in* S.

Proof. See Dieudonné [1, 11.4.1]. □

Theorem 1.2 summarizes the *spectral* theory of compact operators.

An important class of compact operators are the *nuclear* operators, also known as *of trace class*, and the operators of Hilbert-Schmidt type (see [2, 15.4.8]).

We now assume E to be an infinite, *separable* (i.e. with denumerable base) complex Hilbert space.

$u \in \mathscr{L}(E)$ is called $\mathcal{HS}$ (for Hilbert-Schmidt) if there exists a base (e_n) such that $\sum_n \|u(e_n)\|^2$ converges.

If u is $\mathcal{HS}$, then u^* is $\mathcal{HS}$. The number $\|u\|_2 = (\sum_n \|u(e_n)\|^2)^{\frac{1}{2}}$ is independent of the base and $\|u\|_2 = \|u^*\|_2$. The space $\mathscr{L}_2(E)$ of $\mathcal{HS}$-operators is a Banach space with involution.

For $x \in E$ we have $\|u(x)\|^2 \leq \|x\|^2 \cdot \|u\|_2^2$ and $\mathscr{L}_2(E) \subset \mathscr{L}_c(E)$, $\mathcal{HS}$-operators are compact.

$u \in \mathscr{L}(E)$ is called *nuclear* or *of trace class*, if there exists a basis (e_n) such that $\sum_n \|u(e_n)\|$ converges. The number $\|u\|_1 = \sum_n \langle u(e_n)|e_n\rangle$ is called the *trace* of u, it is independent of the base. We have $\|u\|_2 \leq \|u\|_1$ and the space $\mathscr{L}_1(E) \subset \mathscr{L}_2(E)$ of nuclear operators is a Banach space in the $\| \; \|_1$-norm.

References

[1] Jean Dieudonné, *Foundations of Modern Analysis*, Academic Press, 1960.
[2] ———, *Eléments d'Analyse 2*, Gauthiers–Villars, 1968.