ON CALENDAR FORMULAS

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For Ralph

Abstract. This paper deals with calendar formulas for calculating the day of the week and the *Julian* day number as used in Astronomy. It describes my implementation of these formulas in a C program.

Preface

Ten years ago I published an article on *Gauss'* calendar formula [3], which determines the day of the week of the 1^{st} of January of a year A as

$$w \equiv 5 \cdot (A-1) \bmod 4 + 4 \cdot (A-1) \bmod 100 + 6 \cdot (A-1) \bmod 400 \pmod 7$$

where $w = 0$ is *Monday*, $w = 1$ is *Tuesday*, ..., $w = 6$ is *Sunday* [2, XI, 1.].

Here I treat the *Julian* day number j introduced by *John Herschel* in 1849. The *Gauss* formula can be bypassed by $j(y, m, d)$ as $w \equiv j \pmod 7$.

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 version 1.3

1. Leap years through the times

In this article I use the ISO 8601 format for dates: `y-m-d`, in formulas written as (y, m, d). The use of integers for years signifies that I denote 1 BC = 0, etc.

The calendar currently used worldwide is the *Gregorian* calendar, which is derived from the Roman calendar as reformed by *Julius Caesar* in -45 and corrected by *Augustus (Octavian)* in -7. The *Julian* rule prescribes: every four years is a *leap* year. In the early years after the reform by mistake every third year was made a leap year and *Augustus* corrected it (leaving out several leap years). Since year 8 the rule was followed until 1582.

Then a reform by Pope *Gregory XIII* refined this rule by dropping the leap day in years divisible by 100, but not by 400. This adjustment was necessary to keep the *vernal equinox* near the 21^{st} of March. To bring the calendar back into synch the *Gregorian* reform also dropped ten days in October: the last day of the *Julian* calendar was Thursday, 4 October 1582 and was followed by the first day of the *Gregorian* calendar: Friday, 15 October 1582.

Here I treat the dates as if the *Julian* calendar had been in use since eternity until `1582-10-04`, (so called *proleptic* calendar). Keep in mind, though, this is historically not correct before *Caesar*'s time; also, the current numbering of the

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years was invented by *Dionysius Exiguus* in 525 and adopted by the venerable *Bede* in 731 and thereafter only slowly promulgated.

Dates since `1582-10-15` are assumed to follow the *Gregorian* rules forever. The year 1582 was shorter than ordinary years by 10 days, it had only 355 days. Finally the dates `1582-10-5`, `1582-10-6`, ... `1582-10-14` are false, they never existed.

2. Gauss ansatz

In [2, XI, 1.] *Gauss* conceives a numbering of the days by starting at some particular date and linearly extending the numeration to the future. For the 1^{st} of January he writes down a simple formula, reflecting the *Gregorian* leap year rules in a supple way ("geschmeidige Art"). His note is reproduced below as appendix A.2.

We will generalize his approach to arbitrary dates (including the *Julian* era) and extending to the past. We will take a different starting point, which was proposed in 1849 by *John Herschel*: the *Julian* day number, starting at `-4712-1-1`. This is used in *Astronomy*, in particular, where it denotes *noon* of a day.

We consider the group operation of $\mathbf{Z}$ on the set $T \subset \mathbf{Z} \times \{1, \ldots, 12\} \times \{1, \ldots, 31\}$ of dates as in [3, §2.3 shifting dates]:

$$\begin{aligned} \mathbf{Z} \times T &\longrightarrow T \\ (n, t) &\longmapsto n + t \end{aligned}$$

where $n + t$ is defined as t shifted by n days, earlier ($n < 0$) or later ($n > 0$) in time. For a year $a \in \mathbf{Z}$ we let $T_a \subset T$ be the finite set of dates of that year: $t = (y, m, d) \in T_a \Leftrightarrow y = a$. If a is a *leap* year, the cardinal is $|T_a| = 366$, otherwise $|T_a| = 365$, except for $|T_{1582}| = 355$. We can also write $|T_a| = 365 + \ell(a)$ for all $a \neq 1582$, where $\ell(a) = 1$ for a *leap* year and $\ell(a) = 0$ otherwise.

We have obvious maps (*offset*) $x : T_a \longrightarrow \{1, \ldots, 366\}$ counting the days of a year from the first to the last. When $e(m)$ is the number of days in previous months

m	1	2	3	4	5	6	7	8	9	10	11	12
$e(m)$	0	31	59	90	120	151	181	212	243	273	304	334

then for $t \in T_a$, $a \neq 1582$, $x(t) = \begin{cases} d + e(m) & \text{for } m \leq 2 \\ d + e(m) + \ell(a) & \text{for } m > 2 \end{cases}$

Let $z : T \longrightarrow \mathbf{Z}$ be any *linear* map, such that $z(n + t) = n + z(t)$. There is *essentially* one, only depending on the *origin* t_0 such that $z(t_0) = 0$. The choice $t_0 = (-4712, 1, 1)$ will be denoted by j, the *Julian* day number.

For two dates $t, t' \in T$, $t = (a, m, d), t' = (a', m', d')$ and $t < t'$ we have

$$z(t') - z(t) = x(t') - x(t) + \sum_{y=a}^{a'-1} |T_y|$$

and by the above (assuming $y \neq 1582$) we have

$$\sum_{y=a}^{a'-1} |T_y| = (a' - a) \cdot 365 + \sum_{y=a}^{a'-1} \ell(y)$$

The last sum, the number of leap years in the interval $[a, a' - 1]$, is written in the *supple way* by *Gauss* for $a \equiv 1 \bmod 400$

$$\sum_{y=a}^{a'-1} \ell(y) = \left[\frac{a' - a}{4}\right] - \left[\frac{a' - a}{100}\right] + \left[\frac{a' - a}{400}\right]$$

which simply reflects the *Gregorian* leap year rule. Hence

$$z(t') - z(t) = x(t') - x(t) + (a' - a) \cdot 365 + \left[\frac{a' - a}{4}\right] - \left[\frac{a' - a}{100}\right] + \left[\frac{a' - a}{400}\right]$$

The interesting point now is, that this formula is not only true for $t < t'$ but for $t > t'$ as well, as will be derived in the next section.

3. Counting intervals in congruence classes

We prove some simple properties of the *Gauss floor* function $[x]$, which he introduced in 1808 (see [1], p. 459).

Lemma 3.1. *For $m, n \in \mathbf{Z}$, $m > 0$ we have*

$$\left[\frac{-1 - n}{m}\right] = -1 - \left[\frac{n}{m}\right]$$

Proof. We write $n = q \cdot m + r$ with $0 \leq r < m$ such that $q = [n/m]$. Then $0 \leq m - r - 1 < m$ and as $-1 - n = -(q + 1) \cdot m + (m - r - 1)$ we have $[(-1 - n)/m] = -q - 1$. □

Lemma 3.2. *For $a, n, m \in \mathbf{Z}$, $n > 0$, $m > 0$ with $a \equiv 1 \pmod{m}$ we have*

$$|[a, a + n - 1] \cap m\mathbf{Z}| = \left[\frac{n}{m}\right] \tag{1}$$

$$|[a - n, a - 1] \cap m\mathbf{Z}| = -\left[\frac{-n}{m}\right] \tag{2}$$

Proof. For (1): let $n = q \cdot m + r$ with $0 \leq r < m$ and $a = 1 + k \cdot m$. The set

$$[a, a + n - 1] \cap m\mathbf{Z} = \{a - 1 + m, \ldots, a - 1 + q \cdot m = a + n - 1 - r\}$$

has $q = [n/m]$ elements.

For (2): let $n - 1 = q \cdot m + r$ with $0 \leq r < m$ and $a = 1 + k \cdot m$. Then $a - n = k \cdot m - q \cdot m - r = (k - q) \cdot m - r$ and the set

$$[a - n, a - 1] \cap m\mathbf{Z} = \{a - n + r, a - n + r + m, \ldots, a - n + r + q \cdot m = a - 1\}$$

contains $1 + q$ elements, which by lemma 3.1 is $= 1 + [(n - 1)/m] = -[-n/m]$. □

We will apply this to the *leap* years in the *Julian* resp. *Gregorian* era in the interval $a \leq y < a'$ for $a < a'$ and in the interval $a' \leq y < a$ for $a' < a$.

In the *Julian* case this set is $[a, a' - 1] \cap 4\mathbf{Z}$ for $a < a'$ and $[a', a - 1] \cap 4\mathbf{Z}$ for $a' < a$.

In the *Gregorian* case this set is $[a, a' - 1] \cap 4\mathbf{Z} \setminus [a, a' - 1] \cap 100\mathbf{Z} \cup [a, a' - 1] \cap 400\mathbf{Z}$ for $a < a'$ and $[a', a - 1] \cap 4\mathbf{Z} \setminus [a', a - 1] \cap 100\mathbf{Z} \cup [a', a - 1] \cap 400\mathbf{Z}$ for $a' < a$. Remark that the union is *disjoint*.

Corollary 3.3. *Let $t, t' \in T$, $t = (a, m, d), t' = (a', m', d')$.*

In the Julian *era: $t, t' \leq$ 1582-10-04, for $a \equiv 1 \pmod{4}$*

$$z(t') - z(t) = x(t') - x(t) + (a' - a) \cdot 365 + \left[\frac{a' - a}{4}\right] \tag{3}$$

In the Gregorian *era: $t, t' \geq$ 1582-10-15, for $a \equiv 1 \pmod{400}$*

$$z(t') - z(t) = x(t') - x(t) + (a' - a) \cdot 365 + \left[\frac{a' - a}{4}\right] - \left[\frac{a' - a}{100}\right] + \left[\frac{a' - a}{400}\right] \tag{4}$$

Proof. The corollary is clear for $a < a'$. For $a' < a$ we remark that $z(t) - z(t') = x(t) - x(t') + (a - a') \cdot 365 + \sum_{y=a'}^{a-1} \ell(y)$ and applying Lemma 3.2 (2) with $n = a - a'$ we get

$$z(t) - z(t') = x(t) - x(t') + (a - a') \cdot 365 - \left[\frac{a' - a}{4}\right] + \left[\frac{a' - a}{100}\right] - \left[\frac{a' - a}{400}\right]$$

with the last two terms only in the *Gregorian* case. Multiplying by -1 gives the asserted formula. □

4. Programming notes

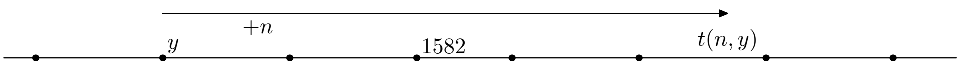
In this section I explain some points of the program `cal.c` in appendix A.1.

There are four functions: `main`, `date`, `leap` and `quot`, the latter three reused from `gauss` [3]. `leap` implements the *leap* function $\ell(y)$ and `quot` the *floor* function $[n/m]$ for $m > 0$. Function `date` needs some explanation.

4.1. Function `date`.

It is called with an *offset* n and by setting y to a *starting* year. Think of it as a *linear* function $t : \mathbf{Z} \times \mathbf{Z} \to T$, $t(n, y) = (a, m, d)$ with $t(k + n, y) = k + t(n, y)$. It is understood that $n = 1$ corresponds to the first day of the year y, $t(1, y) = (y, 1, 1)$, hence offset $n = 0$ corresponds to New Year's Eve, $t(0, y) = (y - 1, 12, 31)$.

Another special point is that I have *not* implemented the exceptional year 1582, which only has 355 days. That is, I have to avoid situations like the following, crossing the critical year. In the figure I start in the *Julian* era and add an offset getting me to the *Gregorian* era.



Similarly, situations have to be avoided starting in the *Gregorian* era and taking us back to the *Julian* era by a *negative* offset.

4.2. Some *Julian* day numbers.

We need to understand some of the constants in the `main` function, gathered in this table

$$\begin{array}{rrr} j(0,12,31) = 1,721,423 & j(1581,12,31) = 2,298,883 & j(1582,10,4) = 2,299,160 \\ j(1582,10,15) = 2,299,161 & j(1582,12,31) = 2,299,238 & j(2000,12,31) = 2,451,910 \end{array}$$

Using (3) with $t' = (-4712, 1, 1)$ and $t = (1, 1, 1)$ we get

$$j(-4712, 1, 1) - j(1, 1, 1) = -4713 \cdot 365 + \left[\frac{-4713}{4}\right] = -1720245 - 1179 = -1721424$$

As by definition $j(-4712, 1, 1) = 0$ we obtain $j(1, 1, 1) = 1721424$.

The next value is $j(1582, 1, 1) = j(1, 1, 1) + 1581 \cdot 365 + [1581/4] = 1721424 + 577065 + 395 = 2298884$. $j(1582, 10, 4) = j(1581, 12, 31) + 4 + e(10) = 2298883 + 4 + 273 = 2299160$. $j(1582, 12, 31) = j(1581, 12, 31) + 355 = 2299238$. Using (4) we have $j(2001, 1, 1) = j(1583, 1, 1) + 418 \cdot 365 + 1 + [418/4] - [418/100] + [418/400] = 2299239 + 152570 + 1 + 104 - 4 + 1 = 2451911$.

4.3. **The `main` function.** The first half of the `main` function does housekeeping and parameter checking.

The following lines

```
if (y < 1582 || y == 1582 && m*100+d < 1005) // Julian era
   j = 1721423 + d + e[m] + (y-1)*365 + quot(y-1,4);
else {                                       // Gregorian era
   x = y - 2001;
   j = 2451910 + d + e[m] + x*365 + quot(x,4) - quot(x,100) + quot(x,400);
}
```

set the *Julian* day number $j = j(t)$ depending on the era, making use of the formulas (3) resp. (4), with the values just calculated.

The next two lines adjust for leap year and add the *offset* from the command line.

```
   if (m>2) j+=leap(y);                       // correction for leap year
   j += n;                                    // add offset
```

Eventually, the correct dates are calculated by calling function `date` either for a *Julian* date starting with 1582, or for a *Gregorian* date starting with year 1583 – to avoid the crossing of the critical year described above.

```
if (j < 2299161) {                     // Julian era
   y=1582;
   date(j-2298883,&y,&m,&d);           // calculate the date
}
else {                                 // Gregorian era
   y=1583;
   date(j-2299238,&y,&m,&d);           // calculate the date
}
```

The remainder of the `main` function is obvious.

Appendix A.

A.1. **The program `cal.c`.** The input syntax is `date [offset]`, where *date* is in ISO 8601 format `year-mm-dd` and *offset* can be any integer. Weeks run from Monday to Sunday, its numbering follows ISO.

The program checks for correct dates under the assumption that

- all dates previous to `1582-10-04` are dates in the proleptic *Julian* calendar
- all dates past `1582-10-15` are dates in the proleptic *Gregorian* calendar
- the dates in the range `1582-10-05` to `1582-10-14` are rejected as incorrect

The program `cal` displays

- the day of the week (Mon, Tue, Wed, Thu, Fri, Sat, Sun)
- the date (*Julian* for $\leq$ `1582-10-04`, *Gregorian* for $\geq$ `1582-10-15`)
- the Julian day number (`J#`)
- the day of the year (`D#`)
- the week of the year (`W#`)

The program operates correctly in the range of dates `-5,877,908-3-15` to `+5,874,898-6-3` corresponding to values of j in the range $-2^{31} - 1 + 2,298,883 \leq j \leq 2^{31} - 1 = 2,147,483,647$. This range originates from the overflow in the type `long int` of j.

Program listing `cal.c`. *//*

```
/* ---------------------------------------------------------------------

   Module:      cal.c

   Description: Calendar formulas

        Input:  year-mm-dd [offset] (ISO 8601 Format)

        Output: day of the week, date, Julian day number,
                day of the year, week of the year

   Example: cal 1777-04-30 +84005
            Mon 2007-04-30 J# 2454221 D# 120 W# 18

   Documentation: http://berndt-schwerdtfeger.de/cal/calj.pdf

   ---------------------------------------------------------------------

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   limitations under the License.

   ---------------------------------------------------------------------*/

#include <stdio.h>
#include <stdlib.h>

// set constants

char *wd[] = {"Mon", "Tue", "Wed", "Thu", "Fri", "Sat", "Sun"};
int  e[13] = {0,0,31,59,90,120,151,181,212,243,273,304,334};

// function prototypes

void date(long,long*,int*,int*);     // offset, year, month, day
int  leap(long);                     // = 1 if year is a leap year
long quot(long,long);                // = [a/b] (corrected: works if a < 0)

// ---------------------------------------------------------------------
// main function
// ---------------------------------------------------------------------

int main(int argc, char *argv[]){

   if (argc == 1 || *argv[1] == '?'){
```

```
      printf("\ncal $Revision: 105 $, $Date: 2011-04-05 20:46:18 +0200 (Di, 05 Apr 2011) $ \n");
      printf("-- Copyright (C) 2010 Berndt E. Schwerdtfeger -- \n\n");
      printf("Input:  year-mm-dd [offset] \n\n");
      printf("Output: {Mon|..|Sun} year-mm-dd J# jjjjjjj D# nnn W# ww\n\n");
      printf("Example: cal 1777-04-30 +84005 \n");
      printf("Mon 2007-04-30 J# 2454221 D# 120 W# 18\n\n");
      return EXIT_SUCCESS;              // end the program
   }                                    // argc > 1 here
   int  w,m,d;
   long x,y,j,n=0;

   sscanf(argv[1],"%ld-%d-%d",&y,&m,&d);

   if (argc == 3)
      sscanf(argv[2],"%ld",&n);         // get the offset
   else if (argc > 3) {
      printf("Too many parameters !\n");
      return EXIT_FAILURE;              // end the program
   }
   if (y==1582 && m==10 && d>4 && d<15) {
      printf("This date is invalid, it did never exist !\n");
      return EXIT_FAILURE;              // end the program
   }
   if (y < 1582 || y == 1582 && m*100+d < 1005) // Julian era
      j = 1721423 + d + e[m] + (y-1)*365 + quot(y-1,4);
   else {                               // Gregorian era
      x = y - 2001;
      j = 2451910 + d + e[m] + x*365 + quot(x,4) - quot(x,100) + quot(x,400);
   }
   if (m>2) j+=leap(y);                 // correction for leap year
   j += n;                              // add offset
   if (j < 2299161) {                   // Julian era
      y=1582;
      date(j-2298883,&y,&m,&d);         // calculate the date
   }
   else {                               // Gregorian era
      y=1583;
      date(j-2299238,&y,&m,&d);         // calculate the date
   }
   n = d + e[m];                        // offset into this year
   if (m > 2)
      n += leap(y);                     // adjust for leap day
   if (y==1582 && n>278) n-=10;         // ten days dropped in 1582
   w = j%7;                             // set weekday
   if (w<0) w+=7;
   printf("%s %ld-%02d-%02d J# %ld ", wd[w],y,m,d,j);
   printf("D# %03d W# %02d\n", n,(n-w+9)/7);
   return EXIT_SUCCESS;                 // exit the program
};

// ------------------------------------------------------------------------
// function definitions
// ------------------------------------------------------------------------

void date(long n, long* y, int* m, int* d){

   int i;
```

```
   while (n > 365 + leap(*y)){          // if offset larger than # of
      n -= 365 + leap(*y);              // ... days in a year
      *y += 1;                          // ... find the correct year
   }
   while (n <= 0){                      // if offset is negative
      *y -= 1;                          // ... find the correct year
      n += 365 + leap(*y);              // ... and offset
   }
   i = leap(*y);                        // adjust for leap day
   *m = 12;
   while (n <= e[*m] + i){              // searching for the month
      *m -= 1;
      if (*m<3) i=0;
   }
   *d = n - e[*m] - i;                  // setting the day
}

// ------------------------------------------------------------------------
int leap(long y){

   int i = 0;
   if (y%4 == 0)
      i = 1;
   if (y > 1582 && y%100 == 0 && y%400 != 0)
      i = 0;
   return i;
}

// ------------------------------------------------------------------------
long quot(long a, long b){

   long x = a/b;
   if (a < 0 )                          // for negative numerator ..
      x -= (a%b != 0);                  // .. if remainder, subtract 1
   return x;
}

//
```

A.2. **Gauss' calendar formula in his Nachlass.** *Gauss* did not publish his formula for finding the day of the week. It appeared posthumously in his *Werke* in 1927 [2, XI, p. 206]. The *bracket* notation for the floor function $[x]$ was not yet used, *Gauss* introduced it in 1808.

Den Wochentag des 1. Januar eines Jahres zu finden.

Handschriftliche Eintragung in: Sammlung astronomischer Tafeln, unter Aufsicht der Kgl. Preußischen Akademie der Wissenschaften, I. Band, Berlin 1776. — 1798 von Gauss erworben.

Bezeichnet man den kleinsten positiven Rest einer Grösse A nach dem Modulus m durch $R : A \pmod m$, so lassen sich alle Vorschriften des Gregorianischen Kalenders auf folgende geschmeidige Art darstellen:

1.

Wenn man die Tage vom 1^{ten} Januar 1701 an zählt, d. i. diesen mit 1, den 2^{ten} mit 2, den 31^{ten} Dec. 1700 mit 0, den 30^{ten} mit -1 etc. bezeichnet, so ist der 1^{te} Januar in irgend einem Jahre A

$$\begin{aligned}= 1 + (A-1701)365 + \frac{1}{4}\big((A-1701) - R:(A-1701) \bmod 4\big)\\ - \frac{1}{100}\big((A-1701) - R:(A-1701) \bmod 100\big)\\ + \frac{1}{400}\big((A-1601) - R:(A-1601) \bmod 400\big)\end{aligned}$$

2.

Die Wochentage Sonntag, Montag, etc. mit $0, 1$ etc. bezeichnet, ist der 1^{te} Januar irgend eines Jahres, qua Wochentag,

$$\left.\begin{aligned}\equiv 6 + A + (2A + 5R:(A-1) \bmod 4)\\ + (3A + 4R:(A-1) \bmod 100)\\ + (A + 2 + 6R:(A-1) \bmod 400)\end{aligned}\right\} \bmod 7.$$

Also von 1601 bis 2000

$$\equiv 6 + 6A + 5R:(A-1) \bmod 4 + 4R:(A-1) \bmod 100.$$

Allgemein

$$\equiv 1 + 5R:(A-1) \bmod 4 + 4R:(A-1) \bmod 100 + 6R:(A-1) \bmod 400.$$

References

[1] Carl Friedrich Gauss, *Untersuchungen über höhere Arithmetik*, 2nd (reprinted), AMS Chelsea Publishing, Providence, 1981.

[2] ———, *Werke* (1863), available at http://gdz.sub.uni-goettingen.de/dms/load/toc/?PID=PPN235957348.

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