INVARIANTS OF CURVES OF SECOND ORDER

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Abstract. *Conics* occur as orbits of massive bodies in a gravitational field, hence they are important in *Astronomy* [5, § 1.2 celestial mechanics]. The *orbital elements* are fundamental invariants of the curve. Here we determine the invariants from the coefficients of the curve.

Preface

Real plane curves $V \subset \mathbb{R}^2$ of second order are defined by a polynomial $Q \in \mathbb{R}[X,Y]$ of $\deg Q = 2$, $V = V(Q)$ is its vanishing set of points (x,y) where $Q(x,y) = 0$. They occur as intersections of planes with cones in space, hence they are called *conic sections* (or *conics*).

Attached to conics are several invariants: *eccentricity* ε, *semi-latus rectum* ℓ, *semi-major axis* a, *focus* F, *directrix* d, ...etc. Determining the invariants of curves of second order

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

from the coefficients a_{ik} of the equation is a well known task. We derive formulas for the invariants following an elementary high school course developed by Freudenthal and Heinrich 50 years ago. This paper explains their approach, omitting examples, exercises and applications to pencils of curves – consult their more detailed German booklets [2–4]. See also Coxeter's *Introduction to Geometry* on conics [1, 8.4 Conics].

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1. Curves of second order

The equation of an algebraic *curve of second order* consists of quadratic and linear terms and an absolute term:

$$Q(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \tag{1.1}$$

To Q we will associate the *symmetric* matrix $M = (a_{ik})$, such that $Q(x,y) = (x,y,1) \cdot M \cdot {}^t(x,y,1)$. For $q \in \mathbb{R}^\times$ the polynomials Q and $Q' = q \cdot Q$ define the *same* curve as the vanishing sets are equal $V(Q') = V(Q)$ (and it may be *empty* $V(Q) = \varnothing$, of course). At least one quadratic term must be present, i.e. $(a_{11}, a_{12}, a_{22}) \neq (0,0,0)$, otherwise we would have a line equation.

We denote by $\mathcal{Q}_2$ the set of all such $Q \mod \mathbb{R}^\times$ and call it the *set $\mathcal{Q}_2$ of curves of second order*; it is a five dimensional object. The group of plane isometries $\mathbf{O}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ having dimension 3 we expect individual conics to be described by 2 fundamental parameters.

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1.1. **Focal generation.** A large part of $\mathcal{Q}_2$ can be described by the concept of *focal generation* developed by Pappus of Alexandria ($\sim 290 - 350$) for conics:

Given a real number $\varepsilon > 0$, a point $F \in \mathbb{R}^2$ and a line $d \subset \mathbb{R}^2$ not containing F, the *locus of points* P whose distance to F equals ε times their distance to d is a *conic*.

If we relax this definition and allow $F \in d$ we get *degenerate* curves : intersecting lines, double lines, points . . . .

The number ε is the *eccentricity*, the point F is the *focus*, the line d is the *directrix*. With $r = \overline{PF}$, $s = \overline{Pd}$ as in figure 1 the definition requires $r = \varepsilon \cdot s$. A conic is called *ellipse* if $\varepsilon < 1$, *parabola* if $\varepsilon = 1$ and *hyperbola* if $\varepsilon > 1$. The names are due to Apollonios of Perga ($\sim 262 - 190$ BC).

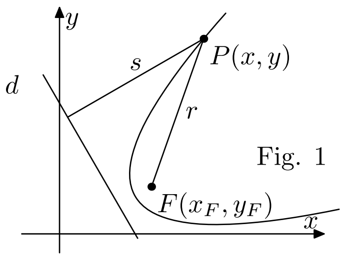


Fig. 1

Let $d = \xi x + \eta y + \zeta = 0$ with $\xi^2 + \eta^2 = 1$ be the equation of the *directrix* in *Hesseform* such that F is on the *positive* side of the directrix: the distance $f = \overline{Fd}$ of the focus F from the directrix d is $d(F) = \xi x_F + \eta y_F + \zeta = f \geq 0$. We obtain the polynomial defining the conic from the equation

$$r^2 = \varepsilon^2 s^2 \tag{1.2}$$

with $d(P) = \xi x + \eta y + \zeta = \pm s$ as (see figure 1)

$$\varepsilon^2(\xi x + \eta y + \zeta)^2 - (x - x_F)^2 - (y - y_F)^2 = 0$$

which rearranged reads

$$\begin{aligned}(\varepsilon^2\xi^2 - 1)x^2 + 2\varepsilon^2\xi\eta xy + (\varepsilon^2\eta^2 - 1)y^2 + 2(\varepsilon^2\xi\zeta + x_F)x + 2(\varepsilon^2\eta\zeta + y_F)y +\\ + \varepsilon^2\zeta^2 - x_F^2 - y_F^2 = 0\end{aligned} \tag{1.3}$$

The coefficients of (1.3) will be written as matrix

$$M(\varepsilon, F, d) = \begin{pmatrix} \varepsilon^2\xi^2 - 1 & \varepsilon^2\xi\eta & \varepsilon^2\xi\zeta + x_F \\ \varepsilon^2\xi\eta & \varepsilon^2\eta^2 - 1 & \varepsilon^2\eta\zeta + y_F \\ \varepsilon^2\xi\zeta + x_F & \varepsilon^2\eta\zeta + y_F & \varepsilon^2\zeta^2 - x_F^2 - y_F^2 \end{pmatrix} \tag{1.4}$$

where you have to keep in mind that we *normalised* the coefficients of the *directrix* by $\xi^2 + \eta^2 = 1$ and $f = \xi x_F + \eta y_F + \zeta > 0$. We call (1.4) the *normalised matrix* of the curve.

These *focally generated* curves define a correspondence

$$\begin{aligned} \mathbb{R}_+^\times \times \mathbb{R}^2 \times \mathbb{L}(\mathbb{R}^2) &\longrightarrow \mathcal{Q}_2 \\ (\varepsilon, F, d) &\longmapsto [M(\varepsilon, F, d)] \end{aligned} \tag{1.5}$$

where the set of lines in the plane is $\mathbb{L}(\mathbb{R}^2) \simeq \mathbb{S}^1 \times \mathbb{R}$. This correspondence is *neither injective* (end of §1.2.2) *nor surjective* (§1.2.3).

1.2. **Fundamental invariants.** When the *directrix* is the y-axis, $d = x = 0$, we have $\xi = 1$, $\eta = \zeta = 0$ and the focal point lies on the x-axis $F = (f, 0)$, and (1.3) simplifies to

$$(1 - \varepsilon^2)x^2 + y^2 - 2fx + f^2 = 0 \tag{1.6}$$

We recognize from (1.6) that the x-axis $y = 0$ is a *symmetry axis* through F orthogonal to the *directrix*, which is called the *major axis*. For $x = f$ the curve equation (1.6) yields $y = \pm\varepsilon f$. We set

$$\ell = \varepsilon f \tag{1.7}$$

it is called *semi-latus rectum* ℓ of the curve.

1.2.1. *Apex equation.* The *major axis* intersects the conic in a point lying between the focal point F and the *directrix*, called the *apex* A. We put the apex A in the origin, keeping the major axis as x-axis and the directrix parallel to the y-axis.

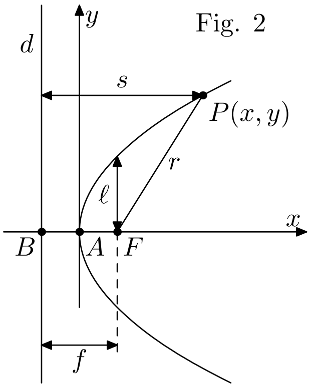


Fig. 2

From figure 2 for $P = A$ we get $r_A = x_F$, $s_A = -x_B$, as $r = \varepsilon s$ we obtain $x_F = -\varepsilon x_B$. We also read that $-x_B + x_F = f = \ell/\varepsilon$. This yields $x_F = \dfrac{\ell}{1+\varepsilon}$ and $x_B = -\dfrac{\ell}{\varepsilon(1+\varepsilon)}$. Now (1.2) implies

$$\left(x - \frac{\ell}{1+\varepsilon}\right)^2 + y^2 = \varepsilon^2 \left(x + \frac{\ell}{\varepsilon(1+\varepsilon)}\right)^2$$

and finally the *apex equation for conics*

$$y^2 = 2\ell x - (1-\varepsilon^2)x^2 \tag{1.8}$$

1.2.2. *Ellipse and hyperbola.* In the cases $\varepsilon \neq 1$ the intersections A_1 and A_2 of the curve with the *major axis* $y = 0$ by (1.6) have the abscissa $x_{A_{1,2}} = \frac{f}{1\pm\varepsilon}$. The intersection points $A_{1,2} = (\frac{f}{1\pm\varepsilon}, 0)$ are called *major apexes* of the curve. The distance $\overline{A_1A_2} = |x_{A_1} - x_{A_2}| = \frac{2\ell}{|1-\varepsilon^2|}$ is the *major axis* of the curve. We set the *semi-major axis* to

$$a = \frac{\ell}{1-\varepsilon^2} \tag{1.9}$$

Remark. In our convention the *hyperbola* has $a < 0$!

The major axis has the *centre*

$$C = \Big(\frac{x_{A_1} + x_{A_2}}{2}, 0\Big) = \Big(\frac{f}{1-\varepsilon^2}, 0\Big) = \Big(\frac{a}{\varepsilon}, 0\Big) \tag{1.10}$$

The *centre* has the *distance* to the *directrix* $+\frac{a}{\varepsilon}$ for the ellipse and $-\frac{a}{\varepsilon}$ for the hyperbola. By translating $x = x' + \frac{f}{1-\varepsilon^2}, y = y'$ (the centre C becomes the origin in the x'-y'-frame) we obtain for (1.6) the form $(1-\varepsilon^2)x'^2 + y'^2 = \ell^2/(1-\varepsilon^2)$ or

$$\frac{x'^2}{\left(\frac{\ell}{1-\varepsilon^2}\right)^2} + \frac{y'^2}{\left(\frac{\ell}{\sqrt{1-\varepsilon^2}}\right)^2} = 1 \tag{1.11}$$

We set the *semi-minor axis* to

$$b = \frac{\ell}{\sqrt{1-\varepsilon^2}} \tag{1.12}$$

b is imaginary for the hyperbola. From (1.11) we recognize *symmetries* of the curve to centre C and to the y'-axis. The latter is called the *minor axis*; its intersections with the curve are called *minor apexes* $B_{1,2}$. The distance $\overline{B_1B_2}$ is called *minor axis* of the curve. By (1.11) we have $y_{B_{1,2}} = \pm b$.

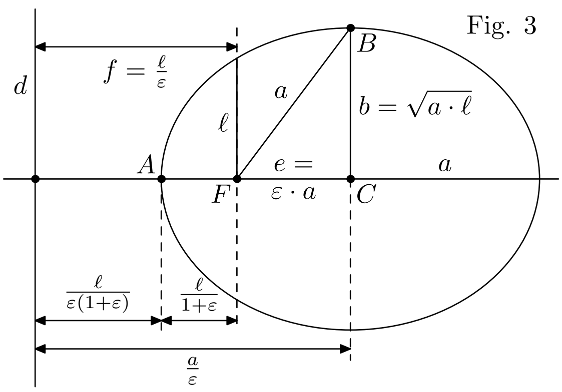


Fig. 3

The *major* and *minor* axes are collectively called the *principal axes* of the curve. In the coordinate system of these axes the common equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.13}$$

of *ellipse* and *hyperbola* is called the *principal axes equation* or also *central equation*; *ellipse* and *hyperbola* are also called *central* curves.

By (1.7), (1.9), (1.10) (see figure 3) we conclude that the distance $\overline{FC} = |x_C - f| = \varepsilon|a|$. We put

$$e = \varepsilon a \tag{1.14}$$

and call e the *linear eccentricity* of the curve ; it is negative for the hyperbola.

We note the following relations

$$b^2 = a\ell \qquad\qquad a^2 = b^2 + e^2 \tag{1.15}$$

Figure 3 depicts the geometrical meaning of all invariants established so far for an *ellipse*.

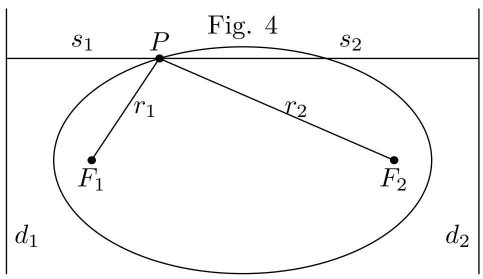
From the *symmetry* to the *minor axis* for ellipse and hyperbola follows the existence of a *second directrix* d_2 and a *second focal point* F_2. The previous directrix will then be noted $d = d_1$, the focus with $F = F_1$, in particular we have $M(\varepsilon, F_1, d_1) = M(\varepsilon, F_2, d_2)$ and the *correspondence* (1.5) is *not injective*.

1.2.3. *Focally not generated curves.* Let us keep ℓ fixed and consider what happens to the ellipse (1.13) when we let $\varepsilon \to 0$. Then $a \to \ell, b \to \ell, e \to 0$ and $f \to \infty$, hence the focus $F \to C$ and the *directrix* disappears to the periphery of the plane. In the limit we get the equation of the *circle* $x^2 + y^2 = \ell^2$ of *radius* ℓ. So, *circles* should be assigned $\varepsilon = 0$ and can't be generated by the recipe in §1.1.

As circles are not in the image of the *correspondence* (1.5) the set $\mathcal{Q}_2^{foc} \subset \mathcal{Q}_2$ of focally generated curves is a proper subset, (1.5) is *not surjective* and there are *focally not generated* curves $\mathcal{Q}_2 - \mathcal{Q}_2^{foc} \neq \varnothing$.

1.2.4. *Focal properties.* For an ellipse the distance of the two *directrices* from each other is $s_1 + s_2$ (fig. 4) and by (1.10) this implies $s_1 + s_2 = 2\frac{a}{\varepsilon}$, hence by $r = \varepsilon s$ we get $r_1 + r_2 = \varepsilon(s_1 + s_2) = 2a$. This relation is often used as alternative *definition for ellipses*.

The *focal poperty for hyperbola* reads $r_1 - r_2 = \pm 2a$ where the $+$ sign is valid for the right branch, the $-$ sign for the left branch (proof in [4, Aufgabe 1.13]).



1.2.5. *Parabola.* As $\varepsilon = 1$ for a parabola, by (1.6) there is only one *symmetry axis* $(y = 0)$. Its *apex* lies at $A(\frac{\ell}{2}, 0)$.

1.3. **Polar coordinates.** Let us put the focus in the origin $F = (0, 0)$, see figure 5, then since $x = r\cos\varphi$ equation (1.2) gives $\big(r - \varepsilon(f + r\cos\varphi)\big)\big(r + \varepsilon(f + r\cos\varphi)\big) = 0$, so $r - \varepsilon(f + r\cos\varphi) = 0$ or $r + \varepsilon(f + r\cos\varphi) = 0$, yielding the two *polar equations*

$$r = \frac{\ell}{1 - \varepsilon\cos\varphi} \qquad\qquad r = -\frac{\ell}{1 + \varepsilon\cos\varphi} \tag{1.16}$$

We emphasise that *polar equations* are not equations of curves of second order, they are only *factors* of a curve equation. This explains the fact that for a hyperbola the *left* hand side of (1.16) is only valid for the *right* branch and the *right* hand side of (1.16) is only valid for the *left* branch. For ellipse and parabola only the first formula in (1.16) delivers curve points, as the second always gives $r < 0$, since $\varepsilon \leq 1$ and $\ell > 0$.

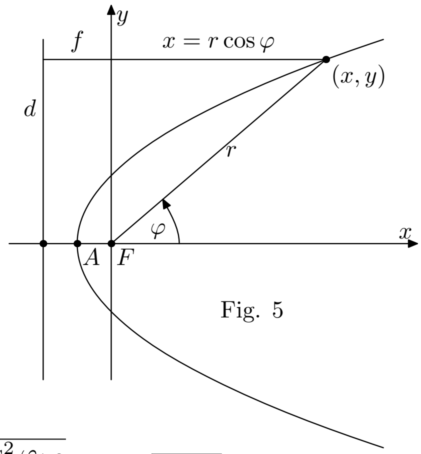


Fig. 5

In case $\varepsilon > 1$ the denominators in (1.8) vanish for $\cos\varphi_{1,2} = \pm\frac{1}{\varepsilon}$, i.e. for focal rays with slopes

$$m_{1,2} = \tan\varphi_{1,2} = \pm\frac{\sqrt{1-\cos^2\varphi_{1,2}}}{\cos\varphi_{1,2}} = \pm\sqrt{\varepsilon^2-1} \tag{1.17}$$

For these arguments $\varphi_{1,2}$ the focal rays show asymptotic behaviour. The parallels to these rays through the *centre* are called *asymptotes* a_1 und a_2 of the hyperbola; they will be generally defined in section §3.2 as *self-conjugate diameters*. They have the equations

$$a_{1,2} = y \mp \sqrt{\varepsilon^2-1}\, x = 0$$

in the system of the axes of the hyperbola. The slopes of the asymptotes are $m_{1,2} = \pm\sqrt{-\frac{\ell}{a}}$, as by (1.9) $\varepsilon^2 - 1 = -\frac{\ell}{a}$. For $\ell \lesseqqgtr -a$ is $m_1 \lesseqqgtr 1$ and $\varepsilon^2 \lesseqqgtr 2$, i.e. $2\varphi_1 \lesseqqgtr \frac{\pi}{2}$; $2\varphi_1$ is the angular segment containing the hyperbola. With $2\varphi_1$ the hyperbola itself is called *acute*, *right-angled* and *obtuse*. The *angular type* of the *asymptotes* of a *hyperbola* is described invariantly at the end of section §2.2.

2. Invariants, eigenvectors and principal axes

The group G of plane isometries consists of the group of *rotations* $\mathbf{O}(2,\mathbb{R})$ and the group of *translations* $\mathbb{R}^2$, the rotations operating on the plane on the *right*, given by $u \cdot \sigma = us + v$ for $\sigma = (s,v) \in G$, $u \in \mathbb{R}^2$. The composition in G is given by $(s,v)\cdot(t,w) = (st, vt+w)$ and G is the semi-direct product $G = \mathbf{O}(2,\mathbb{R}) \ltimes \mathbb{R}^2$. The representation $\rho : G \to \mathbf{GL}(3,\mathbb{R})$, $\rho(s,v) = \left(\begin{smallmatrix} s & 0 \\ v & 1\end{smallmatrix}\right)$ represents G as subgroup of $\mathbf{GL}(3,\mathbb{R})$ where $s \in \mathbf{O}(2,\mathbb{R})$ and $v \in \mathbb{R}^2$. Given a curve $[Q] \in \mathcal{Q}_2$ with matrix $M = (a_{ik})$, its equation (1.1) can be written with $u = (x,y)$ and ${}^t u = \left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right)$ as

$$Q(u) = \begin{pmatrix} u & 1\end{pmatrix}\cdot M \cdot \begin{pmatrix} {}^t u \\ 1\end{pmatrix} = \begin{pmatrix} x & y & 1\end{pmatrix}\cdot\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ 1\end{pmatrix}$$

We are going to find G-invariants from the matrix coefficients of the curve.

2.1. Invariants under movement. An isometric movement $\sigma \in G$ introduces another coordinate system $u' = (x', y')$ in the plane with $u = u'\sigma$, giving the curve an equation $Q'(u') = Q(u)$ with matrix coefficients $M' = (a'_{ik})$, which we are going to explicate now.

$$Q(u) = \begin{pmatrix} u & 1\end{pmatrix}\cdot M \cdot \begin{pmatrix} {}^t u \\ 1\end{pmatrix} = \begin{pmatrix} u' & 1\end{pmatrix}\rho(\sigma)\cdot M \cdot {}^t\rho(\sigma)\begin{pmatrix} {}^t u' \\ 1\end{pmatrix}$$

hence

$$M' = \rho(\sigma)\cdot M \cdot {}^t\rho(\sigma) \tag{2.1}$$

We also consider the *minor matrix* of the *quadratic* terms $M_{33} = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{smallmatrix}\right)$ and we obtain from (2.1) when $\sigma = (s, v)$

$$M'_{33} = sM_{33}{}^t s \tag{2.2}$$

For $s \in \mathbf{O}(2, \mathbb{R})$ we have ${}^t s = s^{-1}$. Thus the relation (2.2) implies that the *trace* $t_{33} = \operatorname{tr} M_{33} = a_{11} + a_{22}$ of M_{33} and its *determinant* $A_{33} = \det M_{33} = a_{11}a_{22} - a_{12}^2$ are *independent of the coordinate system* $t'_{33} = t_{33}$ and $A'_{33} = A_{33}$.

As $\det \rho(\sigma) = \det s$ and $\det {}^t\rho(\sigma) = \det {}^t s$, the relation (2.1) tells us that the *determinant* $D = \det M$ is an *invariant under movement* as well, hence the important relations

$$t'_{33} = t_{33} \qquad\qquad A'_{33} = A_{33} \qquad\qquad D' = D$$

2.2. **Eigenvalues and invariants.** The *invariants under movement* of §2.1 are not invariants of the curve : if we multiply (1.1) by a factor $q \neq 0$ then $Q' = qQ$ and $M' = qM$ and the quantities t_{33}, A_{33} and D behave like

$$t'_{33} = qt_{33} \qquad\qquad A'_{33} = q^2 A_{33} \qquad\qquad D' = q^3 D \tag{2.3}$$

We call these quantities *quasi-invariants* due to their *independence* from the coordinate system and *dependence* on the arbitrary factor q.

For a *focally generated* curve we read its *quasi-invariants* from (1.4) as

$$t_{33} = \varepsilon^2\xi^2 - 1 + \varepsilon^2\eta^2 - 1 = \varepsilon^2 - 2$$

taking into account that $\xi^2 + \eta^2 = 1$, and

$$A_{33} = (\varepsilon^2\xi^2 - 1)(\varepsilon^2\eta^2 - 1) - \varepsilon^4\xi^2\eta^2 = 1 - \varepsilon^2$$

Finally, for the determinant of $M(\varepsilon, F, d)$ we make use of the factorisation

$$M(\varepsilon, F, d) = \begin{pmatrix} \varepsilon\xi & -1 & 0 \\ \varepsilon\eta & 0 & -1 \\ \varepsilon\zeta & x_F & y_F \end{pmatrix} \cdot \begin{pmatrix} \varepsilon\xi & \varepsilon\eta & \varepsilon\zeta \\ 1 & 0 & -x_F \\ 0 & 1 & -y_F \end{pmatrix}$$

so

$$D = \begin{vmatrix} \varepsilon\xi & -1 & 0 \\ \varepsilon\eta & 0 & -1 \\ \varepsilon\zeta & x_F & y_F \end{vmatrix} \cdot \begin{vmatrix} \varepsilon\xi & \varepsilon\eta & \varepsilon\zeta \\ 1 & 0 & -x_F \\ 0 & 1 & -y_F \end{vmatrix} = \varepsilon^2(\xi x_F + \eta y_F + \zeta)^2 = \varepsilon^2 f^2$$

hence by (1.7) $D = \ell^2$. We summarize the result for an arbitrary matrix of a *focally generated* curve taking (2.3) into account

$$t_{33} = q \cdot (\varepsilon^2 - 2) \qquad\qquad A_{33} = q^2 \cdot (1 - \varepsilon^2) \qquad\qquad D = q^3 \cdot \ell^2 \tag{2.4}$$

The sign $\operatorname{sig} A_{33}$ is even an *invariant*: an *ellipse* has $A_{33} > 0$, a *parabola* has $A_{33} = 0$ and a *hyperbola* has $A_{33} < 0$.

From these *quasi-invariants* we will calculate all *invariants* for focally generated curves. We want to determine q. By a simple *change of sign* (replacing Q by $-Q$) we achieve that $q > 0$. From now on we postulate this *sign convention*

$$\begin{aligned} &\text{If } A_{33} \geq 0 \text{ choose } t_{33} < 0 \\ &\text{If } A_{33} < 0 \text{ choose } D > 0 \end{aligned} \tag{2.5}$$

which ensures $q > 0$.

Remark. In the *elliptical* case $A_{33} > 0$ it may happen by this *sign convention* that $D < 0$, see [3, §2.4, p.19]. These curves have *no rational points* $V(Q) = \varnothing$ as can be seen by a *principal axes transformation* (section §2.5, equation (2.39)) and we *neglect* these curves in the sequel. We also *ignore* the *degenerate cases* $D = 0$ here. In the *parabolic* case the *sign convention* implies $D > 0$, see §2.4 below.

The *characteristic polynomial* of the matrix M_{33} (or the curve) is

$$\begin{vmatrix} a_{11}-X & a_{12} \\ a_{12} & a_{22}-X \end{vmatrix} = X^2 - t_{33}X + A_{33} = (X-\lambda_1)(X-\lambda_2), \tag{2.6}$$

its roots are the *eigenvalues* λ_1, λ_2 which are *real*, as M_{33} is *symmetric*[1]. We have

$$\lambda_1 + \lambda_2 = t_{33} \qquad \lambda_1 \cdot \lambda_2 = A_{33} \tag{2.7}$$

As the roots are per se *indistinguishable* we make the arbitrary choice

$$\lambda_1 \geq \lambda_2 \tag{2.8}$$

The discriminant of (2.6) is using (2.4)

$$t_{33}^2 - 4A_{33} = q^2(\varepsilon^2-2)^2 - 4q^2(1-\varepsilon^2) = q^2\varepsilon^4$$

So the eigenvalues $\lambda_{1,2} = (t_{33} \pm \sqrt{t_{33}^2 - 4A_{33}})/2$ are (remark: we make use of $q > 0$)

$$\lambda_1 = q(\varepsilon^2 - 1) \qquad \lambda_2 = -q \tag{2.9}$$

We have

$$\frac{\lambda_1}{\lambda_2} = 1 - \varepsilon^2 \tag{2.10}$$

and the list of all invariants from quasi-invariants follows:

$$\varepsilon = \sqrt{\frac{\lambda_1 - \lambda_2}{-\lambda_2}} \tag{2.11}$$

and by (2.4) and (2.9)

$$\ell = \sqrt{\frac{D}{(-\lambda_2)^3}} \tag{2.12}$$

For the *distance* f of the *directrix* from the corresponding *focus* F we have by (1.7) $\ell = \varepsilon f$, hence by (2.11) and (2.12)

$$f = -\frac{1}{\lambda_2}\sqrt{\frac{D}{\lambda_1 - \lambda_2}} \tag{2.13}$$

For $A_{33} \neq 0$ we get from (1.9) and (1.12) accordingly

$$a = -\frac{1}{\lambda_1}\sqrt{\frac{D}{-\lambda_2}} \tag{2.14}$$

$$b = -\frac{1}{\lambda_2}\sqrt{\frac{D}{-\lambda_1}} \tag{2.15}$$

By (1.14) and (2.7) holds

$$e = \frac{1}{A_{33}}\sqrt{(\lambda_1 - \lambda_2)D} \tag{2.16}$$

$$\frac{a}{\varepsilon} = -\frac{1}{\lambda_1}\sqrt{\frac{D}{\lambda_1 - \lambda_2}} \tag{2.17}$$

Finally we describe *invariantly* the *angular type* of the *asymptotes* of a hyperbola. We may take its quasi-invariants from the *axes* or *central equation* (1.13), with (1.15) $b^2 = a\ell < 0$! By multiplying (1.13) by $a^2\ell$ we obtain $\ell x^2 + ay^2 - a^2\ell = 0$, with matrix $\begin{pmatrix} \ell & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a^2\ell \end{pmatrix}$ and quasi-invariants $t_{33} = \ell + a$, $A_{33} = a\ell < 0$ and

[1] the discriminant of (2.6) is $t_{33}^2 - 4A_{33} = (a_{11} - a_{22})^2 + 4a_{12}^2 \geq 0$

$D = (-a)^3\ell^2 > 0$. Hence: $\ell \lesseqgtr -a$ for $t_{33} \lesseqgtr 0$ which by §1.3 signifies that the *hyperbola* is *acute* for $t_{33} < 0$, *right-angled* for $t_{33} = 0$ and *obtuse* for $t_{33} > 0$.

2.3. **Eigenvectors and directrices.** For a curve of second order given by a polynomial Q as in (1.1) with matrix M we define linear polynomials g_1, g_2, g_3 as follows

$$\begin{aligned} g_1 &= a_{11}x + a_{12}y + a_{13} \\ g_2 &= a_{12}x + a_{22}y + a_{23} \\ g_3 &= a_{13}x + a_{23}y + a_{33} \end{aligned} \tag{2.18}$$

By this definition we have

$$Q = g_1x + g_2y + g_3 \tag{2.19}$$

The assumption $(a_{11}, a_{12}, a_{22}) \neq (0, 0, 0)$ does not imply that the polynomials (2.18) really define lines. For example when $\varepsilon = 1$ in (1.6) we have $(a_{11}, a_{12}) = (0, 0)$.

Given a *focus* $F = (x_F, y_F)$ the associated *directrix* d is given by the equations

$$d = g_1x_F + g_2y_F + g_3 = 0 \tag{2.20}$$

$$d = g_1(F)x + g_2(F)y + g_3(F) = 0 \tag{2.21}$$

Proof. Obviously, it suffices to assume $M = M(\varepsilon, F, d)$. From (1.4) we get

$$g_1x_F + g_2y_F + g_3 = \varepsilon^2(\xi x_F + \eta y_F + \zeta)(\xi x + \eta y + \zeta) = \varepsilon^2 f(\xi x + \eta y + \zeta)$$

which for $f \neq 0$ implies (2.20). By *symmetry* of M (2.20) implies (2.21). □

We now determine a *directrix* $d = \xi x + \eta y + \zeta = 0$ with $\xi^2 + \eta^2 = 1$ (in *Hesseform*) and a *focus* $F = (x_F, y_F)$ from a matrix $M = (a_{ik})$ which satisfies (2.5). If the curve is *focally generated* it is of the form $M = q \cdot M(\varepsilon, F, d)$ with $q = -\lambda_2 > 0$. We start with the coefficients ξ and η of the *directrix*. From (1.4) with (2.9) and (2.10) we have

$$\varepsilon^2\xi^2 = 1 + \frac{a_{11}}{-\lambda_2} = \frac{a_{11} - \lambda_2}{-\lambda_2} \qquad \varepsilon^2\xi\eta = \frac{a_{12}}{-\lambda_2}$$
$$\varepsilon^2\eta^2 = 1 + \frac{a_{22}}{-\lambda_2} = \frac{a_{22} - \lambda_2}{-\lambda_2}$$

which cannot be solved for $\varepsilon = 0$. Let us consider $\varepsilon = 0$: by (2.10) this implies $\lambda_1 = \lambda_2$, the *characteristic polynomial* has a double root, its discriminant vanishes $t_{33}^2 - 4A_{33} = (a_{11} - a_{22})^2 + 4a_{12}^2 = 0$, hence $a_{11} = a_{22}$ and $a_{12} = 0$. As $a_{11} \neq 0$ this implies a *circle* equation

$$\left(x + \frac{a_{13}}{a_{11}}\right)^2 + \left(y + \frac{a_{23}}{a_{11}}\right)^2 = \frac{a_{13}^2 + a_{23}^2 - a_{11}a_{33}}{a_{11}^2} \tag{2.22}$$

Then again *circles* around (x_0, y_0) of *radius* r have the well known equation

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

implying the properties $a_{11} = a_{22}$ and $a_{12} = 0$, which means $\varepsilon = 0$. Hence *circles* are characterised by the equivalent conditions

$$\varepsilon = 0 \iff a_{11} = a_{22} \text{ and } a_{12} = 0 \tag{2.23}$$

Since $A_{33} > 0$ they are incorporated to the *elliptic* types.

Now consider the case $\varepsilon \neq 0$, by (2.10) we have

$$\xi^2 = \frac{a_{11} - \lambda_2}{\lambda_1 - \lambda_2} \qquad \eta^2 = \frac{a_{22} - \lambda_2}{\lambda_1 - \lambda_2} \qquad \xi\eta = \frac{a_{12}}{\lambda_1 - \lambda_2} \tag{2.24}$$

As λ_2 is a root of (2.6) the product $(a_{11}-\lambda_2)(a_{22}-\lambda_2)=a_{12}^2\geq 0$. Otherwise the sum $a_{11}-\lambda_2+a_{22}-\lambda_2=t_{33}-2\lambda_2=\lambda_1-\lambda_2\geq 0$ by (2.7) and (2.8), hence

$$a_{11}-\lambda_2\geq 0 \qquad\qquad a_{22}-\lambda_2\geq 0$$

The sign of the square roots in (2.24) has to be chosen such that $\operatorname{sig}\xi\eta=\operatorname{sig}a_{12}$:

$$\sqrt{a_{11}-\lambda_2}\cdot\sqrt{a_{22}-\lambda_2}=a_{12} \tag{2.25}$$

The *unit vector* $E=(\xi,\eta)$ parallel to the *major axis* and the *perpendicular* vector $E^{\perp}=(-\eta,\xi)$ parallel to the *directrix* (or to the *minor axis*) are the *eigenvectors* to the *eigenvalues* λ_1 and λ_2 respectively (verify by using (2.24) and (2.25)):

$$E\cdot M_{33}=\lambda_1 E \qquad\qquad E^{\perp}\cdot M_{33}=\lambda_2 E^{\perp} \tag{2.26}$$

From (1.4) we get

$$x_F=-\varepsilon^2\xi\zeta+\frac{a_{13}}{-\lambda_2} \qquad\qquad y_F=-\varepsilon^2\eta\zeta+\frac{a_{23}}{-\lambda_2} \tag{2.27}$$

whereby the existence of the *focus* is played back to the quantity ζ. Again by (1.4)

$$\frac{a_{33}}{-\lambda_2}=\varepsilon^2\zeta^2-x_F^2-y_F^2$$

which yields the quadratic equation for ζ

$$\varepsilon^2\zeta^2-x_F^2-y_F^2+\frac{a_{33}}{\lambda_2}=0$$

into which we put the result of (2.27) yielding

$$\varepsilon^2\zeta^2-(\varepsilon^2\xi\zeta+\frac{a_{13}}{\lambda_2})^2-(\varepsilon^2\eta\zeta+\frac{a_{23}}{\lambda_2})^2+\frac{a_{33}}{\lambda_2}=0$$

giving us eventually (eliminating ε^2 by (2.10))

$$\lambda_1(\lambda_1-\lambda_2)\zeta^2-2(\lambda_1-\lambda_2)(\xi a_{13}+\eta a_{23})\zeta+a_{13}^2+a_{23}^2-a_{33}\lambda_2=0 \tag{2.28}$$

We calculate the discriminant Δ of (2.28)

$$\begin{aligned}\Delta&=4(\lambda_1-\lambda_2)^2(\xi a_{13}+\eta a_{23})^2-4\lambda_1(\lambda_1-\lambda_2)(a_{13}^2+a_{23}^2-a_{33}\lambda_2)=\\&=4(\lambda_1-\lambda_2)\theta\end{aligned}$$

with

$$\begin{aligned}\theta&=(\lambda_1-\lambda_2)\xi^2a_{13}^2+2(\lambda_1-\lambda_2)\xi\eta a_{13}a_{23}+(\lambda_1-\lambda_2)\eta^2a_{23}^2-\\&-\lambda_1a_{13}^2-\lambda_1a_{23}^2+a_{33}\lambda_1\lambda_2\end{aligned}$$

using (2.7) and (2.24) and rearranging terms

$$\theta=(a_{11}-\lambda_2-\lambda_1)a_{13}^2+2a_{12}a_{13}a_{23}+(a_{22}-\lambda_2-\lambda_1)a_{23}^2+a_{33}A_{33}$$

using (2.7) again

$$\begin{aligned}\theta&=-a_{22}a_{13}^2+2a_{12}a_{13}a_{23}-a_{11}a_{23}^2+a_{33}A_{33}=\\&=a_{13}(a_{12}a_{23}-a_{13}a_{22})+a_{23}(a_{12}a_{13}-a_{11}a_{23})+a_{33}A_{33}=\\&=a_{13}A_{13}+a_{23}A_{23}+a_{33}A_{33}=D\end{aligned}$$

hence $\Delta=4(\lambda_1-\lambda_2)D$.

This decides the sign of the discriminant: in the *hyperbolic* case we always get real solutions. In the *elliptic* case we see that $D<0$ yields no real solutions to ζ – we *neglect* these cases for good reason.

In the *parabolic* case (2.28) is linear, which is treated in §2.4. For now assume $A_{33} \neq 0$, hence $\lambda_1 \neq 0$. The solutions of the quadratic equation (2.28) using the results of (2.24) are

$$\zeta_{1,2} = \frac{1}{\lambda_1\sqrt{\lambda_1 - \lambda_2}}\Big(a_{13}\sqrt{a_{11} - \lambda_2} + a_{23}\sqrt{a_{22} - \lambda_2} \pm \sqrt{D}\Big) \tag{2.29}$$

where the sign of the roots have to comply with the sign rule (2.25). Removing the denominator $\lambda_1\sqrt{\lambda_1 - \lambda_2}$ the equations of the *directrices* become

$$\begin{aligned}
\lambda_1\sqrt{a_{11} - \lambda_2}x + \lambda_1\sqrt{a_{22} - \lambda_2}y + a_{13}\sqrt{a_{11} - \lambda_2} + a_{23}\sqrt{a_{22} - \lambda_2} \pm \sqrt{D} = 0\\
\lambda_1(a_{11} - \lambda_2)x + a_{12}\lambda_1 y + a_{13}(a_{11} - \lambda_2) + a_{12}a_{23} \pm \sqrt{(a_{11} - \lambda_2)D} = 0\\
a_{12}\lambda_1 x + \lambda_1(a_{22} - \lambda_2)y + a_{12}a_{13} + a_{23}(a_{22} - \lambda_2) \pm \sqrt{(a_{22} - \lambda_2)D} = 0
\end{aligned} \tag{2.30}$$

The second equation results by multiplication with $\sqrt{a_{11} - \lambda_2}$, the last by multiplication with $\sqrt{a_{22} - \lambda_2}$. Here we made use of (2.24), (2.25).

We have the simple relationships for focal points resp. apexes with the *centre* C

$$F_{1,2} = C \pm eE \qquad A_{1,2} = C \pm aE \qquad B_{1,2} = C \pm bE^{\perp} \tag{2.31}$$

From (2.31) we obtain a formula for the *centre* $C = (x_0, y_0) = \frac{1}{2}(F_1 + F_2)$. By (2.27) we have

$$x_0 = -\varepsilon^2\xi\zeta_0 + \frac{a_{13}}{-\lambda_2} \qquad\qquad y_0 = -\varepsilon^2\eta\zeta_0 + \frac{a_{23}}{-\lambda_2}$$

where we have put $\zeta_0 = \frac{1}{2}(\zeta_1 + \zeta_2)$, which by (2.29) is

$$\zeta_0 = \frac{1}{\lambda_1\sqrt{\lambda_1 - \lambda_2}}\Big(a_{13}\sqrt{a_{11} - \lambda_2} + a_{23}\sqrt{a_{22} - \lambda_2}\Big)$$

Putting it together (using (2.25))

$$\begin{aligned}
x_0 &= \frac{\lambda_1 - \lambda_2}{\lambda_2}\frac{\sqrt{a_{11} - \lambda_2}}{\sqrt{\lambda_1 - \lambda_2}}\frac{1}{\lambda_1\sqrt{\lambda_1 - \lambda_2}}\Big(a_{13}\sqrt{a_{11} - \lambda_2} + a_{23}\sqrt{a_{22} - \lambda_2}\Big) - \frac{a_{13}}{\lambda_2} =\\
&= \frac{1}{\lambda_1\lambda_2}(a_{13}(a_{11} - \lambda_2 - \lambda_1) + a_{23}a_{12}) = \frac{1}{A_{33}}(-a_{13}a_{22} + a_{23}a_{12}) = \frac{A_{13}}{A_{33}}
\end{aligned}$$

Similarly

$$\begin{aligned}
y_0 &= \frac{\lambda_1 - \lambda_2}{\lambda_2}\frac{\sqrt{a_{22} - \lambda_2}}{\sqrt{\lambda_1 - \lambda_2}}\frac{1}{\lambda_1\sqrt{\lambda_1 - \lambda_2}}\Big(a_{13}\sqrt{a_{11} - \lambda_2} + a_{23}\sqrt{a_{22} - \lambda_2}\Big) - \frac{a_{23}}{\lambda_2} =\\
&= \frac{1}{\lambda_1\lambda_2}(a_{13}a_{12} + a_{23}(a_{22} - \lambda_2 - \lambda_1)) = \frac{1}{A_{33}}(a_{13}a_{12} - a_{23}a_{11}) = \frac{A_{23}}{A_{33}}
\end{aligned}$$

This derivation does not work for *circles* as they are not focally generated, but because of (2.23) circles have $A_{13} = -a_{11}a_{13}, A_{23} = -a_{11}a_{23}$ and $A_{33} = a_{11}^2$ such that $A_{13}/A_{33} = -a_{13}/a_{11}$ and $A_{23}/A_{33} = -a_{23}/a_{11}$ which by (2.22) shows the following *centre formula* holds for *circles* as well

$$C = \Big(\frac{A_{13}}{A_{33}}, \frac{A_{23}}{A_{33}}\Big) \tag{2.32}$$

By (2.31) with (2.16) and (2.32), respecting the sign rule (2.25) we get

$$A_{33}F_{1,2} = (A_{13} \pm \sqrt{(a_{11} - \lambda_2)D}, A_{23} \pm \sqrt{(a_{22} - \lambda_2)D})$$

2.4. **Eigenvector in parabolic case.** In the parabolic case we have $\lambda_1 = 0, \lambda_2 = t_{33}, A_{33} = 0$ and (2.24) reads

$$\xi^2 = \frac{a_{22}}{t_{33}} \qquad \eta^2 = \frac{a_{11}}{t_{33}} \qquad \xi\eta = \frac{a_{12}}{-t_{33}} \tag{2.33}$$

The *linear* equation (2.28) becomes

$$2t_{33}(\xi a_{13} + \eta a_{23})\zeta + a_{13}^2 + a_{23}^2 - a_{33}t_{33} = 0$$

hence

$$\zeta = \frac{a_{33}t_{33} - a_{13}^2 - a_{23}^2}{2t_{33}(\xi a_{13} + \eta a_{23})}$$

The numerator is $a_{22}a_{33} - a_{23}^2 + a_{11}a_{33} - a_{13}^2 = A_{11} + A_{22} = T_{33}$, the corresponding trace of the *adjugate* matrix (A_{ik}) of cofactors, similar to t_{33} of the matrix (a_{ik}) of the curve. Mutliplying the *Hesseform* of the *directrix* $d = \xi x + \eta y + \zeta$ with the denominator of ζ gives us

$$2t_{33}(\xi a_{13} + \eta a_{23})d = 2t_{33}(\xi^2 a_{13} + \xi\eta a_{23})x + 2t_{33}(\xi\eta a_{13} + \eta^2 a_{23})y + T_{33}$$

using (2.33)

$$\begin{aligned} &= 2(a_{22}a_{13} - a_{12}a_{23})x + 2(-a_{12}a_{13} + a_{11}a_{23})y + T_{33} = \\ &= -2A_{13}x - 2A_{23}y + T_{33} \end{aligned}$$

and the equation of the *directrix* becomes

$$2A_{13}x + 2A_{23}y - T_{33} = 0 \tag{2.34}$$

We derive other formulas for ξ, η, ζ by making use of $A_{33} = 0$. Using the *adjugate* matrix $\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & 0 \end{pmatrix}$ and making use of the fact that the adjugate of (A_{ik}) is the matrix $(a_{ik} \cdot D)$ we have in particular

$$a_{11}D = -A_{23}^2 \qquad a_{12}D = A_{13}A_{23} \qquad a_{22}D = -A_{13}^2 \tag{2.35}$$

In passing we note that the *sign convention* (2.5) for the parabola implies $D > 0$, since $t_{33}D = -A_{13}^2 - A_{23}^2 < 0$.

Now from (2.33) with (2.35) we obtain

$$\xi = \frac{A_{13}}{\sqrt{-t_{33}D}} \qquad \eta = \frac{A_{23}}{\sqrt{-t_{33}D}} \qquad \zeta = \frac{-T_{33}}{2\sqrt{-t_{33}D}} \tag{2.36}$$

In the parabolic case (2.27) reads with the *eigenvector* $E = (\xi, \eta)$

$$F = -\zeta E - \frac{1}{t_{33}}(a_{13}, a_{23})$$

Now (2.36) yields

$$F = -\frac{1}{t_{33}}(a_{13} + \frac{T_{33}}{2D}A_{13}, a_{23} + \frac{T_{33}}{2D}A_{23}) \tag{2.37}$$

By section §1.2.5 we know that $A = F - \frac{\ell}{2}E$, hence by (2.12) and (2.36) we conclude

$$A = F - \frac{(A_{13}, A_{23})}{2t_{33}^2} \tag{2.38}$$

2.5. **Transformation to principal axes.** By an isometric transformation we can move a curve of second order to a position in which the equation has a most simple form. This will be explicated now.

2.5.1. *Axes equation of central curves ($A_{33} \neq 0$).* We move the curve such that its *centre* lies in the *origin* of the coordinate system and one of its axes becomes the x-axis. Then by (1.13) the new equation of the curve has the form $a'_{11}x^2+a'_{22}y^2+a'_{33} = 0$. From its associated matrix $\begin{pmatrix} a'_{11} & 0 & 0 \\ 0 & a'_{22} & 0 \\ 0 & 0 & a'_{33} \end{pmatrix}$ we obtain $t'_{33} = a'_{11} + a'_{22} = t_{33} = a_{11} + a_{22}$, $A'_{33} = a'_{11}a'_{22} = A_{33} = \lambda_1\lambda_2$ and $D' = a'_{11}a'_{22}a'_{33} = D$, i.e. $A_{33}a'_{33} = D$, hence $a'_{33} = \frac{D}{A_{33}}$. Identifying λ_1 with a'_{11}, hence λ_2 with a'_{22}, the transformed equation becomes

$$\lambda_1 x^2 + \lambda_2 y^2 + \frac{D}{A_{33}} = 0 \tag{2.39}$$

Transforming the original equation (1.1) into (2.39) is called *principal axes transformation* which we could perform *without* calculating the isometry. The x-axis is the *major axis*. Interchanging λ_1 and λ_2 in (2.39) makes the x-axis the *minor axis*.

2.5.2. *Axes equation of parabola ($A_{33} = 0$).* The similar approach for the parabola moving the *apex* into the origin and the *principal axis* to the x-axis requires by (1.8) the new equation of the form $a'_{22}y^2 + 2a'_{13}x = 0$, its matrix $\begin{pmatrix} 0 & 0 & a'_{13} \\ 0 & a'_{22} & 0 \\ a'_{13} & 0 & 0 \end{pmatrix}$ yields $t'_{33} = a'_{22} = t_{33} < 0$, $D' = -(a'_{13})^2 a'_{22} = -(a'_{13})^2 t_{33} = D > 0$, hence the equation

$$y^2 = \pm 2\sqrt{\frac{D}{(-t_{33})^3}} \cdot x \tag{2.40}$$

is the *axis equation* of the parabola where the x-axis is the *principal axis*.

2.6. **Equations of principal axes.** From the knowledge of the *eigenvectors* E, $E^{\perp}$ we can directly determine the equations of the *major* and *minor axis* without a *principal axes transformation*.

The principle we make use of is the following. Linear equations in the direction of $E = (\xi, \eta)$ are of the form $-\eta x + \xi y = (x, y) \cdot {}^tE^{\perp} = c$, since these lines are parallel to the line $-\eta x + \xi y = 0$ which is satisfied by vectors $(x, y) \in \mathbb{R}E$. We may replace E or $E^{\perp}$ by any appropriate multiple λE. Similarly the lines perpendicular to E are of the form $(x, y) \cdot {}^tE = \xi x + \eta y = c$. Instead we may as well take $\xi g_1 + \eta g_2$ since $(\xi a_{11} + \eta a_{12})x + (\xi a_{12} + \eta a_{22})y = \lambda_1(\xi x + \eta y)$ by (2.26).

We note the following formulas for multiples of the *eigenvectors*, consequences of (2.24) and the definitions $E = (\xi, \eta), E^{\perp} = (-\eta, \xi)$

$$(\lambda_1 - \lambda_2)\xi E = (a_{11} - \lambda_2, a_{12}) \qquad (\lambda_1 - \lambda_2)\eta E = (a_{12}, a_{22} - \lambda_2) \tag{2.41}$$

$$(\lambda_1 - \lambda_2)\xi E^{\perp} = -(a_{12}, a_{22} - \lambda_1) \qquad (\lambda_1 - \lambda_2)\eta E^{\perp} = (a_{11} - \lambda_1, a_{12}) \tag{2.42}$$

Let us consider the case of *central* curves. As $\det\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{smallmatrix}\right) = A_{33} \neq 0$, the equations $g_1 = 0$ and $g_2 = 0$ define lines that are not *parallel*, hence they *intersect*, in fact at the *centre* C: by the development rules of determinants we have

$$a_{11}A_{13} + a_{12}A_{23} + a_{13}A_{33} = 0 = a_{12}A_{13} + a_{22}A_{23} + a_{23}A_{33} \tag{2.43}$$

and after dividing by A_{33} this reads $g_1(C) = 0 = g_2(C)$, using (2.32). Consider the line equations

$$\begin{aligned} m &= (a_{11} - \lambda_1)g_1 + a_{12}g_2 = 0 \\ m &= a_{12}g_1 + (a_{22} - \lambda_1)g_2 = 0 \end{aligned} \tag{2.44}$$

We already know $m(C) = 0$, so m passes through the *centre*. By (2.42) the line points in the direction of the major axis, hence it is the *major axis*.

For the *minor axis* taking the coefficients from the *eigenvectors* (2.41) we get

$$n = (a_{11} - \lambda_2)g_1 + a_{12}g_2 = 0 \qquad (2.45)$$
$$n = a_{12}g_1 + (a_{22} - \lambda_2)g_2 = 0$$

Similar equations by using the centre $C = (x_0, y_0)$ are

$$m = (a_{11} - \lambda_1)(x - x_0) + a_{12}(y - y_0) = 0 \qquad (2.46)$$
$$m = a_{12}(x - x_0) + (a_{22} - \lambda_1)(y - y_0) = 0$$

$$n = (a_{11} - \lambda_2)(x - x_0) + a_{12}(y - y_0) = 0 \qquad (2.47)$$
$$n = a_{12}(x - x_0) + (a_{22} - \lambda_2)(y - y_0) = 0$$

For the *parabola* we use the same approach and use the *eigenvector* (2.42) for the line coefficients. We use the *focus* F to verify the equations for the *principal axis* in the *parabolic* case

$$m = a_{11}g_1 + a_{12}g_2 = 0 \qquad (2.48)$$
$$m = a_{12}g_1 + a_{22}g_2 = 0$$

By (2.37) we see that F satisfies $m(F) = 0$ using the relations (2.43) with $A_{33} = 0$.

Remark. The equations (2.48) are the same as (2.44) in the *parabolic* case since $\lambda_1 = 0$.

From (2.48) directly follows for the *parabolic* case

$$m = a_{11}x + a_{12}y + \frac{a_{11}a_{13} + a_{12}a_{23}}{t_{33}} = 0 \qquad (2.49)$$
$$m = a_{12}x + a_{22}y + \frac{a_{12}a_{13} + a_{22}a_{23}}{t_{33}} = 0$$

3. Diameters, tangents and polars

3.1. **Diameters and midpoints of parallel chords.**

Any line through the *centre* of a *central* curve is called a *diameter* and the pencil

$$d = g_1 + \gamma g_2 = 0 \qquad (3.1)$$

is the *pencil of diameters* or *diameter pencil* with parameter γ. The *principal axes* are *diameters* and contained in the pencil. We extend this notion to the *parabolic* case $A_{33} = 0$; hence the *parabolic diameter pencil* consists of *parallel* lines.

Let P, P' be two points on a curve satisfying equations $Q(x, y) = Q(x'y') = 0$. We consider the chord $\overline{PP'}$, its midpoint $M = ((x + x')/2, (y + y')/2)$ and the slope of the chord $s = \frac{y'-y}{x'-x}$, see figure 6. The difference $Q(x', y') - Q(x, y)$ is

$$a_{11}(x'-x)(x'+x)+2a_{12}(x'y'-xy)+a_{22}(y'-y)(y'+y)+2a_{13}(x'-x)+2a_{23}(y'-y) = 0$$

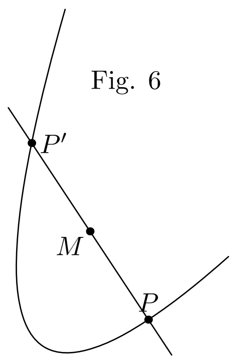
Dividing by $2(x' - x)$ and using $\frac{x'y'-xy}{x'-x} = y_M + sx_M$ we get $a_{11}x_M + a_{12}(y_M + sx_M) + a_{22}sy_M + a_{13} + a_{23}s = 0$, i.e.

$$g_1(M) + sg_2(M) = 0$$

which keeping s constant signifies that the midpoints of all chords of slope s lie on the *diameter* with parameter $\gamma = s$

$$d = g_1 + sg_2 = 0 \qquad (3.2)$$

Fig. 6



which details to $(a_{11}+sa_{12})x+(a_{12}+sa_{22})y+a_{13}+sa_{23}=0$ such that the *diameter slope* is

$$\sigma = -\frac{a_{11}+a_{12}s}{a_{12}+a_{22}s}$$

Hence the *permutability* of σ, s ensues:

$$s = -\frac{a_{11}+a_{12}\sigma}{a_{12}+a_{22}\sigma}$$

From each of the two equations flows the relation – putting $\gamma_1=\sigma$ and $\gamma_2=s$ (or vice versa) –

$$a_{22}\gamma_1\gamma_2+a_{12}(\gamma_1+\gamma_2)+a_{11}=0 \tag{3.3}$$

Two slopes γ_1 and γ_2 are called *conjugate* if they satisfy (3.3).

Remark. In the *parabolic* case the diameter slope $\sigma=-\frac{a_{11}}{a_{12}}=-\frac{a_{12}}{a_{22}}$ is independent of s. See §3.3 for *conjugacy* in the *parabolic* case.

For *central* curves diameters of *conjugate* slopes are called *conjugate diameters*. Conjugate diameters mutually bisect the parallel chords; the *tangents* in the end-points of one diameter are parallel to the conjugate diameter (figure 7). As a central curve is symmetric to major and minor axis the *principal* axes are conjugate diameters.

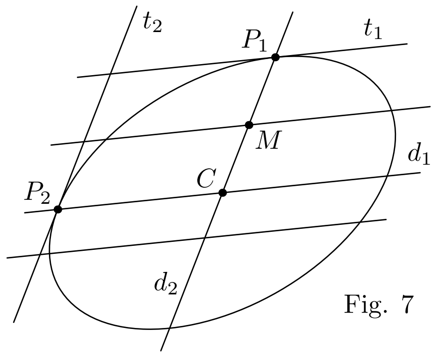


Fig. 7

3.2. **Asymptotes of hyperbola.** The *asymptotes* of a hyperbola pass through the centre and hence are *diameters*, contained in the *diameter pencil* (3.1). The *asymptotic slopes* have an interesting property: they are *self-conjugate*.

Let α be a slope which is *conjugate to itself*. Then by (3.3) with $\gamma_1=\gamma_2=\alpha$ we have

$$a_{22}\alpha^2+2a_{12}\alpha+a_{11}=0 \tag{3.4}$$

For $a_{22}\neq 0$ we obtain two slopes

$$\alpha_{1,2}=\frac{1}{a_{22}}(-a_{12}\pm\sqrt{-A_{33}}) \tag{3.5}$$

which are real in the *hyperbolic* case $A_{33}<0$. For $a_{12}=0$ in particular the curve has the equation $b^2(x-x_0)^2+a^2(y-y_0)^2-a^2b^2=0$ with $a_{11}=b^2<0$ and $a_{22}=a^2$ and (3.5) gives with $A_{33}=a_{11}a_{22}$

$$\alpha_{1,2}=\pm\sqrt{-\frac{a_{11}}{a_{22}}}=\pm\sqrt{-\frac{b^2}{a^2}}=\pm\sqrt{\frac{e^2-a^2}{a^2}}=\pm\sqrt{\varepsilon^2-1}$$

which by (1.17) is the slope of the asymptotes. Hence in this case the *self-conjugate* diameters are the *asymptotes* of the hyperbola.

Hence we call the *self-conjugate diameters* the *asymptotes* of the hyperbola, in agreement with the earlier definition in §1.3.

When we write (3.4) in the form $a_{11}\left(\frac{1}{\alpha}\right)^2+2a_{12}\frac{1}{\alpha}+a_{22}=0$, then we obtain in case $a_{22}=0$

$$\frac{1}{\alpha_1}=0 \text{ and } \alpha_2=-\frac{a_{11}}{2a_{12}}$$

In case $a_{22} \neq 0$ the *asymptotes* have the equations

$$a_{1,2} = a_{22}g_1 + (-a_{12} \pm \sqrt{-A_{33}})g_2 = 0 \tag{3.6}$$

and in case $a_{22} = 0$

$$a_1 = g_2 = 0 \text{ and } a_2 = 2a_{12}g_1 - a_{11}g_2 = 0 \tag{3.7}$$

The constant a_{33} is not involved in the equations of the *asymptotes*. Hence by adding a constant to Q we can obtain a *degenerate* curve which is reducible: it factors into the asymptotes $a_1 \cdot a_2$. This will be made explicit now.

Let $a_{22} \neq 0$ then $(-a_{12} + \sqrt{-A_{33}})(-a_{12} - \sqrt{-A_{33}}) = a_{12}^2 + A_{33} = a_{11}a_{22}$ and (3.6) yields $a_1a_2 = a_{22}^2g_1^2 - 2a_{12}a_{22}g_1g_2 + a_{11}a_{22}g_2^2$, i.e. for $a_{22} \neq 0$

$$\begin{aligned}\frac{a_1a_2}{a_{22}} &= g_1(a_{22}g_1 - a_{12}g_2) + g_2(-a_{12}g_1 + a_{11}g_2) = \\ &= g_1(A_{33}x - A_{13}) + g_2(A_{33}y - A_{23}) = \\ &= A_{33}(g_1x + g_2y + g_3) - (g_1A_{13} + g_2A_{23} + g_3A_{33}) = \\ &= A_{33}Q - D\end{aligned}$$

by (2.19) and $g_1A_{13} + g_2A_{23} + g_3A_{33} = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} = D$, i.e. $a_1a_2 \equiv a_{22}(A_{33}Q - D)$.

If $a_{22} = 0$ then $D = 2a_{12}a_{13}a_{23} - a_{11}a_{23}^2 - a_{12}^2a_{33}$ and $A_{33} = -a_{12}^2$; (3.7) yields $a_1a_2 = a_{12}^2(a_{11}x^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y) + 2a_{12}a_{13}a_{23} - a_{11}a_{23}^2$, i.e. $a_1a_2 = a_{12}^2(Q - a_{33}) + D + a_{12}^2a_{33} = a_{12}^2Q + D = -(A_{33}Q - D)$.

Hence the polynomial $A_{33}Q - D$ is reducible factoring into a_1a_2 in all cases.

3.3. **Conjugacy for parabolas and tangent equation.** In the *parabolic* case either $a_{11} \neq 0$ or $a_{22} \neq 0$. A *parabolic diameter* may be written by (2.49)

$$\begin{aligned} d &= a_{11}x + a_{12}y + c = 0 && \text{if } a_{11} \neq 0 \\ d &= a_{12}x + a_{22}y + c = 0 && \text{if } a_{22} \neq 0 \end{aligned}$$

Comparing $a_{11}Q$ with d^2 we see in case $a_{11} \neq 0$

$$\begin{aligned} a_{11}Q &= a_{11}^2x^2 + 2a_{11}a_{12}xy + a_{11}a_{22}y^2 + 2a_{11}a_{13}x + 2a_{11}a_{23}y + a_{11}a_{33} \\ d^2 &= a_{11}^2x^2 + 2a_{11}a_{12}xy + a_{12}^2y^2 + 2a_{11}cx + 2a_{12}cy + c^2 \end{aligned}$$

respectively $a_{22}Q$ with d^2 in case $a_{22} \neq 0$ that

$$\begin{aligned} a_{22}Q &= a_{22}a_{11}x^2 + 2a_{22}a_{12}xy + a_{22}^2y^2 + 2a_{22}a_{13}x + 2a_{22}a_{23}y + a_{22}a_{33} \\ d^2 &= a_{12}^2x^2 + 2a_{12}a_{22}xy + a_{22}^2y^2 + 2a_{12}cx + 2a_{22}cy + c^2 \end{aligned}$$

and as $a_{11}a_{22} = a_{12}^2$ the difference $a_{11}Q - d^2$ resp. $a_{22}Q - d^2$ is linear

$$\begin{aligned} t = a_{11}Q - d^2 &= 2a_{11}(a_{13} - c)x + 2(a_{11}a_{23} - a_{12}c)y + a_{11}a_{33} - c^2 \\ t = a_{22}Q - d^2 &= 2(a_{22}a_{13} - a_{12}c)x + 2a_{22}(a_{23} - c)y + a_{22}a_{33} - c^2 \end{aligned} \tag{3.8}$$

The line $t = 0$ is not parallel to $d = 0$ as by (2.35) the determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{11}(a_{13} - c) & a_{11}a_{23} - a_{12}c \end{vmatrix} = -a_{11}A_{23} \neq 0 \qquad \text{for } a_{11} \neq 0$$

$$\begin{vmatrix} a_{12} & a_{22} \\ a_{22}a_{13} - a_{12}c & a_{22}(a_{23} - c) \end{vmatrix} = a_{22}A_{13} \neq 0 \qquad \text{for } a_{22} \neq 0$$

The *intersection* point T of d and t is a point on the curve, as $d(T) = 0 = t(T)$ implies immediately that $Q(T) = 0$. Hence T is the end point of the diameter d and t is the *tangent* to the curve in the point T and d bisects the chords parallel to t. Therefore t is called the *conjugate tangent* to d and d is called the *conjugate diameter* to t. With $T = (x_T, y_T)$ we eliminate c from $d(T) = 0$ and obtain

$$c = -a_{11}x_T - a_{12}y_T \quad \text{if } a_{11} \neq 0 \quad \text{resp. } c = -a_{12}x_T - a_{22}y_T \quad \text{if } a_{22} \neq 0$$

As T is a curve point we also have $Q(T) = 0$ which translates to

$$c^2 = a_{11}^2x_T^2 + 2a_{11}a_{12}x_Ty_T + a_{11}a_{22}y_T^2 = -a_{11}(2a_{13}x_T + 2a_{23}y_T + a_{33}) \quad \text{if } a_{11} \neq 0$$
$$c^2 = a_{11}a_{22}x_T^2 + 2a_{12}a_{22}x_Ty_T + a_{22}^2y_T^2 = -a_{22}(2a_{13}x_T + 2a_{23}y_T + a_{33}) \quad \text{if } a_{22} \neq 0$$

We put this into the line equation for the tangent (3.8) and get for the case $a_{11} \neq 0$

$$\begin{aligned} t &= 2a_{11}(a_{13} + a_{11}x_T + a_{12}y_T)x + 2(a_{11}a_{23} + a_{12}(a_{11}x_T + a_{12}y_T))y + a_{11}a_{33} + \\ &\quad + a_{11}(2a_{13}x_T + 2a_{23}y_T + a_{33}) = \\ &= 2a_{11}\big(a_{11}x_Tx + a_{12}(y_Tx + x_Ty) + a_{22}y_Ty + a_{13}(x + x_T) + a_{23}(y + y_T) + a_{33}\big) \end{aligned}$$

The case $a_{22} \neq 0$ being similar we eventually obtain the *tangent equation*

$$t = a_{11}x_Tx + a_{12}(y_Tx + x_Ty) + a_{22}y_Ty + a_{13}(x + x_T) + a_{23}(y + y_T) + a_{33} = 0 \tag{3.9}$$

Other variants to write (3.9) are

$$\begin{aligned} t &= g_1(T)x + g_2(T)y + g_3(T) = 0 \\ t &= g_1x_T + g_2y_T + g_3 = 0 \end{aligned} \tag{3.10}$$

Remark. In section §3.5, formula (3.13) we will see, that for all *curves of second order* the *tangent equations* are given by (3.9), (3.10). The equation (3.9) is said to arise from the equation (1.1) of a curve of second order by *separation*.

The equation of a parabola can be determined from a *diameter* and a *conjugate tangent* as follows

$$Q = d^2 + \kappa t = 0$$

The constant κ can be determined by a given point on the curve.

3.4. **Geometric significance of g_1, g_2.** The *diameter* $d = g_1 + sg_2 = 0$ *conjugate* to *slope* s can in case $s \neq 0$ be written in form $d = \frac{1}{s}g_1 + g_2 = 0$.

If $s = 0$ the conjugate diameter is $d = g_1 = 0$ and the chords bisected by g_1 are parallel to the x-axis, i.e. on g_1 lies (if existent) the highest *summit point* S and the lowest *valley point* V of the curve (figure 8).

If $\frac{1}{s} = 0$ the diameter $d = g_2$ bisects the chords parallel to the y-axis, i.e. on g_2 lies (if existent) the farthest *left point* L respectively the farthest *right point* R of the curve.

The points L, R, S, V are *extremal points*. Their existence depends on the position of the curve with respect to the coordinate system. I will clarify the situation now with a detailed analysis.

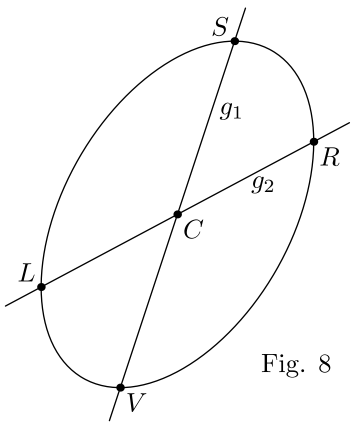


Fig. 8

By (2.18), (2.19) we have

$$\begin{aligned} a_{11}Q - g_1^2 &= a_{11}(g_1x + g_2y + g_3) - (a_{11}x + a_{12}y + a_{13})g_1 = \\ &= a_{11}(g_2y + g_3) - (a_{12}y + a_{13})g_1 = \\ &= (a_{11}g_2 - a_{12}g_1)y + (a_{11}g_3 - a_{13}g_1) = \\ &= (A_{33}y - A_{23})y + (-A_{23}y + A_{22}) \end{aligned}$$

hence the relation

$$a_{11}Q - g_1^2 = A_{33}y^2 - 2A_{23}y + A_{22} \tag{3.11}$$

Similarly we have

$$a_{22}Q - g_2^2 = A_{33}x^2 - 2A_{13}x + A_{11} \tag{3.12}$$

At points S, V the left hand of (3.11) vanishes, hence yielding the ordinate $y_{S,V}$; the coordinate $x_{S,V}$ is calculated by $g_1(S) = 0 = g_1(V)$. For points L, R the left hand of (3.12) vanishes, hence we obtain $x_{L,R}$ and the other coordinate $y_{L,R}$ is found by $g_2(L) = 0 = g_2(R)$.

3.4.1. *Parabola* ($A_{33} = 0$). In this case the equations (3.11) and (3.12) are linear and a parabola has *either* S *or* V, as well as *either* L *or* R; we will have to find out which case occurs. We remark that any line parallel to the principal axis intersects a parabola once; it suffices to see this in the case of equation (1.8) where obviously $y = c$ has a unique solution for x.

A *special* situation arises when the principal axis is parallel to the coordinate axes, that is when $a_{12} = 0$, then either $a_{11} = 0$ and g_1 does not exist or $a_{22} = 0$ and g_2 does not exist. For $a_{11} = 0$ we have L or R but there is no S nor V, for $a_{22} = 0$ we have S or V but there is no L nor R.

In the *general* situation ($a_{12} \neq 0$) both $g_1 = 0$ and $g_2 = 0$ define lines. Let us note y_1 the root of (3.11) and x_2 the root of (3.12); denote by (x_1, y_1) the intersection of g_1 with the parabola and (x_2, y_2) the intersection of g_2 with the parabola. Hence the system of equations

$$\begin{aligned} -2A_{23}y_1 + A_{22} &= 0 \qquad & -2A_{13}x_2 + A_{11} &= 0 \\ a_{11}x_1 + a_{12}y_1 + a_{13} &= 0 \qquad & a_{12}x_2 + a_{22}y_2 + a_{23} &= 0 \end{aligned}$$

The point (x_1, y_1) is either S or V, by definition of being *summit* or *valley* we have the case that $S = (x_1, y_1) \iff y_1 > y_2$ and $V = (x_1, y_1) \iff y_1 < y_2$.

Similarly holds $L = (x_2, y_2) \iff x_2 < x_1$ and $R = (x_2, y_2) \iff x_2 > x_1$. Therefore we calculate the differences $x_2 - x_1$ and $y_1 - y_2$. We calculate

$$\begin{aligned} x_2 - x_1 &= \frac{A_{11}}{2A_{13}} + \frac{a_{12}A_{22} + 2a_{13}A_{23}}{2a_{11}A_{23}} = \frac{a_{11}A_{11}A_{23} + 2a_{13}A_{13}A_{23} + a_{12}A_{13}A_{22}}{2a_{11}A_{13}A_{23}} = \\ &= \frac{(a_{11}A_{11} + a_{13}A_{13})A_{23} + (a_{13}A_{23} + a_{12}A_{22})A_{13}}{2a_{11}A_{13}A_{23}} = \end{aligned}$$

by (2.43) and (2.35)

$$= \frac{(D - a_{12}A_{12})A_{23} - a_{11}A_{12}A_{13}}{2a_{11}A_{13}A_{23}} = \frac{D}{2a_{11}A_{13}} = \frac{a_{22}D}{2a_{12}^2A_{13}} = -\frac{A_{13}}{2a_{12}^2}$$

Hence for the *parabola* we have the result

$$A_{13} < 0 \implies R = (x_2, y_2) \qquad\qquad A_{13} > 0 \implies L = (x_2, y_2)$$

The case $A_{13} = 0$ signifies $a_{22} = 0$ and there is *neither* L *nor* R as we have seen. The similar calculation for $y_1 - y_2$ yields $y_1 - y_2 = -A_{23}/2a_{12}^2$, hence the result

$$A_{23} < 0 \implies S = (x_1, y_1) \qquad\qquad A_{23} > 0 \implies V = (x_1, y_1)$$

Again $A_{23} = 0$ has been dealt with as this implies $a_{11} = 0$ and there is *neither* S *nor* V in this case.

3.4.2. *Central curves ($A_{33} \neq 0$).* For geometrical reasons an *ellipse* has four different intersection points with g_1 and g_2: all *extremal points* exist.

But for a *hyperbola* the existence of extremal points is not evident. In any case by *symmetry* to the *centre* the existence of V implies the existence of S and vice versa. The same reasoning applies to L, R. Hence if the discriminant of (3.11) vanishes, the line g_1 cannot intersect the hyperbola; similarly for g_2, if the discriminant of (3.12) vanishes there is no intersection with the hyperbola. Now the discriminant of (3.11) is $A_{23}^2 - A_{22}A_{33} = -a_{11}D$ and the discriminant of (3.12) is $A_{13}^2 - A_{11}A_{33} = -a_{22}D$. We conclude that the *extremal points* S and V exist if $a_{11}D < 0$ and L and R exist if $a_{22}D < 0$.

3.5. **Tangents and polars.** Let T be a curve point and t the *tangent* to the curve in T of slope s, so the equation of the tangent is $y - y_T = s(x - x_T)$. On the other hand by section §3.1, (3.2) the *diameter* through T has the equation $d = g_1 + sg_2$. As $d(T) = 0$ we have $g_1(T) + sg_2(T) = 0$, hence $s = -g_1(T)/g_2(T)$ if $g_2(T) \neq 0$. Hence the *tangent equation* is $t = g_1(T)x + g_2(T)y - g_1(T)x_T - g_2(T)y_T = 0$. As T is a curve point $Q(T) = g_1(T)x_T + g_2(T)y_T + g_3(T) = 0$. Hence the *tangent equation* can be written

$$t = g_1(T)x + g_2(T)y + g_3(T) = 0 \tag{3.13}$$

This equation implies the variants (3.9) and (3.10).

Remark. The case $g_2(T) = 0$ is contained in (3.13).

Let $P_0 = (x_0, y_0)$ be a point in the plane and let t_1, t_2 be the *tangents* from P_0 to the curve with contact points T_1 and T_2 as in figure 9. The tangents have equations given by (3.13) and their intersection point is P_0, hence the two equations $t_1(P_0) = 0 = t_2(P_0)$. This will now be interpreted as: the contact points T_1, T_2 lie on the line

$$p_0 = g_1x_0 + g_2y_0 + g_3 = 0 \tag{3.14}$$

which is the *contact secant* and is called *polar* p_0 to the *pole* P_0.

Similar to the *tangent equations* the equation of the *polar* can be put in the equivalent forms

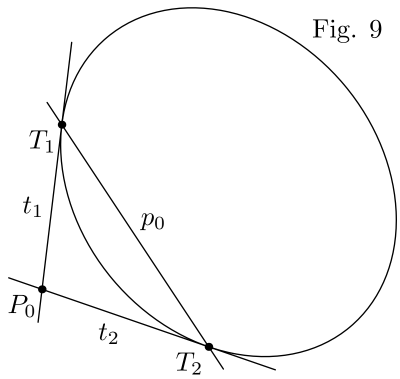


Fig. 9

$$p_0 = g_1(P_0)x + g_2(P_0)y + g_3(P_0) = 0 \tag{3.15}$$

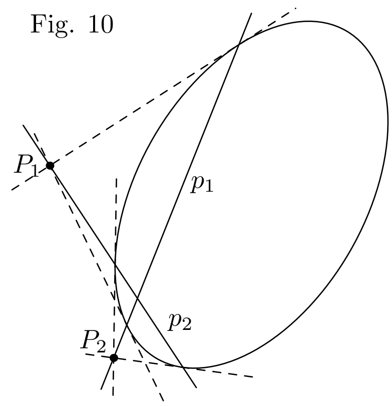
and

$$p_0 = a_{11}xx_0 + a_{12}(xy_0 + yx_0) + a_{22}yy_0 + a_{13}(x + x_0) + a_{23}(y + y_0) + a_{33} = 0 \tag{3.16}$$

Thus we obtain the *tangents* from P_0 to the curve by intersecting the curve with the *polar* p_0, determine its intersection points T_1, T_2 and at last write down the *tangent equations* t_1 and t_2 by (3.13) (or (3.9), (3.10)).

Christian von Staudt (1798–1867) interpreted (3.14) as mapping: by the matrix $M = (a_{ik})$ he mapped any point $P_0 = (x_0, y_0)$ via (3.14) to a *line* p_0: $P_0 \mapsto p_0$. To this map is associated the *distinguished* set of all points which are contained in their image line, i.e. $p_0(P_0) = 0$; but $Q(x_0, y_0) = p_0(P_0)$ and hence his point set is a *curve of second order*, which is Staudt's definition of curves of second order. Thus his definition views the map as the primary object. There are maps without points of coincidence, the associated curve is void, without rational points.

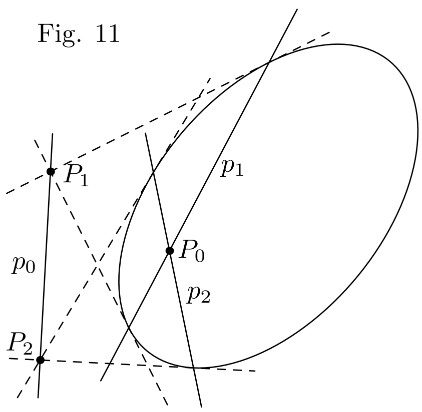
Fig. 10



Let $p_1, P_1 = (x_1, y_1)$ and $p_2, P_2 = (x_2, y_2)$ be two pairs of *polar* and *pole*, see figure 10. We have $p_1 = g_1x_1 + g_2y_1 + g_3$ by (3.14) and $p_2 = g_1(P_2)x + g_2(P_2)y + g_3(P_2)$ by (3.15). Hence $p_1(P_2) = g_1(P_2)x_1 + g_2(P_2)y_1 + g_3(P_2) = p_2(P_1)$. In particular the vanishing of the left and right hand sides are equivalent; hence P_1 *lies on* p_2 is equivalent to P_2 *lies on* p_1.

Construction of the polar p_0 to the pole P_0 *inside* a curve: when p_1 and p_2 are arbitrary secants through P_0, viewed as *polars*, the application of figure 10 delivers the insight of the correctness of the construction of figure 11.

Fig. 11



When a line p rotates around P_0 its pole P runs along the line p_0.

The *directrix* d is the *polar* to its corresponding *focus* as pole, see (2.20) in §2.3.

The line g_3 is the polar of the origin (clear).

Remark. Replacing in the equation of a curve of second order the term $a_{11}x^2 = a_{11}xx$ by $a_{11}xx_0$, $2a_{13}x = a_{13}(x + x)$ by $a_{13}(x + x_0)$ and continuing with the remaining terms accordingly passes the equation of the curve into the equation of the *polar* (3.16). This process has been called *separation* earlier (see §3.3).

3.6. **Line coordinates.** We tackle the task to find the pole $P_0 = (x_0, y_0)$ to a given polar $p = ux + vy + w = 0$ for a curve with matrix $M = (a_{ik})$. Let p_0 be given by (3.15), then $p = \kappa p_0$ for some $\kappa \neq 0$, hence by comparing the coefficients $u = \kappa g_1(P_0), v = \kappa g_2(P_0), w = \kappa g_3(P_0)$, i.e. we have to solve the following system:

$$
\begin{aligned}
a_{11}\kappa x_0 + a_{12}\kappa y_0 + a_{13}\kappa &= u \\
a_{12}\kappa x_0 + a_{22}\kappa y_0 + a_{23}\kappa &= v \\
a_{13}\kappa x_0 + a_{23}\kappa y_0 + a_{33}\kappa &= w
\end{aligned}
$$

CRAMER's rule yields (case $D \neq 0$):

$$\kappa x_0 = \frac{\begin{vmatrix} u & a_{12} & a_{13} \\ v & a_{22} & a_{23} \\ w & a_{23} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}} = \frac{A_{11}u + A_{12}v + A_{13}w}{D}$$

$$\kappa y_0 = \dots = \frac{A_{12}u + A_{22}v + A_{23}w}{D}$$

$$\kappa = \dots = \frac{A_{13}u + A_{23}v + A_{33}w}{D}$$

or as well

$$\begin{aligned} \kappa D x_0 &= uA_{11} + vA_{12} + wA_{13} \\ \kappa D y_0 &= uA_{12} + vA_{22} + wA_{23} \\ \kappa D &= uA_{13} + vA_{23} + wA_{33} \end{aligned} \tag{3.17}$$

Hence the pole is

$$P_0 = \left(\frac{A_{11}u + A_{12}v + A_{13}w}{A_{13}u + A_{23}v + A_{33}w}, \frac{A_{12}u + A_{22}v + A_{23}w}{A_{13}u + A_{23}v + A_{33}w}\right)$$

Let us query which lines $t = ux + vy + w = 0$ are *tangents* to a given curve with matrix $M = (a_{ik})$. By multiplying the left hand sides of (3.17) respectively with u, v, w and subsequently adding up, the left hand side becomes $\kappa D t(P_0) = 0$ as the tangents are characterised by their incidence with its poles. Hence the right hand side vanishes and we have achieved our goal:

$$A_{11}u^2 + 2A_{12}uv + A_{22}v^2 + 2A_{13}uw + 2A_{23}vw + A_{33}w^2 = 0 \tag{3.18}$$

is the equation of the curve in so called *line coordinates* u, v, w. This equation arises by exchange of the coefficients a_{ik} with the *cofactors* A_{ik} in place of the a_{ik}. *Tangents* are those lines $t = ux + vy + w = 0$, whose coefficient triple (u, v, w) satisfy (3.18).

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